16/8/22

CW8502 - OPERATIONS RESEARCH

UNIT - 1 LINEAR MODELS (9)

The phase of an operation research study - Linear programming - graphical method - simplex algorithm - Duality formulation - sensitivity Analysis. n

UNIT - 2 TRANSPORTATION AND NETWORK MODELS (9) n

Transportation, Assignment models - Travelling salesman problem - Network models - shortest route - Minimum spanning tree - Maximum flow models - project Networks - CPM and PERT networks - Critical path scheduling - sequencing models ons MT)

UNIT - 3 INVENTORY MODELS (9)

Inventory models - Economic order quantity models - quantity Discount models - stochastic models - multiproduct models - Inventory control models in practice.

UNIT - 4 QUEUING MODELS (9)

Queuing models - Queuing systems and structures - Rotation parameter - single server and multiserver models - poisson input - Exponential Service - consantrate service - infiring em) population - stimulation.

UNIT-5 DECISION MODELS

Decision Models - Game Theory - 2 /. 0 sum games - graphical solution - Algebraic solution, linear programming solution, Replacement models - Models based on service life - Economic life - single / multi variable search technique - Dyna programming - simple problems

TEXT BOOKS :

1. Hiller and Liberman "Operations Research", Holden day, 2005.
2. Taha. H.A "OR", 6th edition, Prentice Hall of India, 2003.

REFERENCES :

1. Basria, M.J.
Jarvis and Hsearalih.
("Linear programming and Networks") johnwilly 200

2. Budnick, F.S "Principles of operational Research and Management" Richard D Irwin 1990.

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UNIT - I

LINEAR MODELS

Introduction to Operations Research (OR):

OR is the study of optimization technique (Maximize or minimize) which means maximize the profit and minimize the Total cost. It applies Decision Theory.

Different teams had to do research on military operations in order to invent available resources so as to applied the decisions objective. Hence the nomenclature operations Research or Resource Management Technique (RMT)

Uses of OR:

OR is useful to solve

1. Resource Allocation problems.
2. Inventory control problems.
3. Maintanence and Replacement problems.
4. Sequencing and Scheduling problems.
5. Assignment problems.
6. Transportation problems
7. Shortest route problems (Travelling Salesman problem)
8. Marketing Management problems
9. Finance Management Problems.

10. Production, planning and control problems

11. Design Problems

12. Queuing problems

Main phases of OR:

1. Formulation of the problem:

Identifying the objective, the decision variables involved and constraints that arise involving the decision variables.

2. Construction of Mathematical model:

Expressing the measure of effectiveness which may be total profit to be optimized by mathematical.

Representing constraints, Budgets, raw materials, etc... by the needs of mathematical equations or inequalities

3. Solving the model constructed:

Determining the solution by iterative or monte carlo or analytic method depending upon the models.

4. Controlling and updating:

The structural relationship

between the variables may change all these are determined in updating.

The value of one or more control variables vary relationship between the variables undergoes a change. Therefore controls must be established to indicate the limit within which the model and its solution can be considered as reliable. This is called controlling.

5. Testing the model: (ie, validating the model)

From the past available data are experienced whether the model gives the solution which can be used in practice.

6. Implementation:

Using the solution to achieve the desired goal.

Linear Programming (LP):

An analysis of problems represented in linear function with no. of variables that is to be optimized when subjected no. of restraints in the form of inequalities is called linear Programming (LP)

Mathematical for a LPP: (Linear programming problem)

Optimise (Max or Min) $Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_i x_j$

Subject to constraints :

$$\Sigma a_{ij} x_j \ (\leq = \geq) \ b_j \ , \ i = 1, 2, \ldots\ldots m$$

$$\text{and } x_j \geq 0 \ ; \ j = 1, 2, \ldots\ldots n$$

Where $x_1, x_2, \ldots\ldots x_n$ be 'n' decision Variables.

Formation of LPP :

1. A firm uses lathes, milling machines and grind machine to produce 2 machine parts. Table give below represents the machining type required for each part, machining types available machines and profit on each part.

Type of Machines	Machining time required for the Machine part (min)		Max time ava per week (min)
	A	B	
Lathes	12	6	3000
Milling Machine	4	10	2000
Grinding Machine	2	3	900
Profit per Unit	₹ 40	₹ 100	

formulate the problem so that the no. of par I & II to be manufactured per week to maxim the profit

soln: The LPP is

$\text{Max}(Z) = 40x_1 + 100x_2$

Sub to

$12x_1 + 6x_2 \leq 3000$

$4x_1 + 10x_2 \leq 2000$

$2x_1 + 3x_2 \leq 900$

where $x_1, x_2 \geq 0$.

2. A firm is engaged in producing two products P_1 & P_2 and each unit of product P_1 requires 2 kg of raw materials and 4 labour hrs for processing whereas each unit of product P_2 requires 5 kg of raw materials and 3 labour hrs of the same type. Every week the firm has availability of 50 kg of raw material and 160 hrs (labour). 1 unit of product p_1 sold and earn profit ₹ 20. And unit of product p_2 and sold gives ₹ 30 as a profit. Formulate this problem has linear programming to determine has to how many units of each of the product should be produced so that the firm can earn max profit. Assume all the units sold in the market?

The complete form of LPP as follows.

$\text{Max } z = 20x_1 + 30x_2$

subject to constraints

$9x_1 + 5x_2 \le 50$ (Raw materials)

$4x_1 + 3x_2 \le 60$ (Labour hrs)

and $x_1, x_2 \ge 0$

Solution to LPP :

There are two methods to solve LPP. They are

1. Graphical Method (involves only two variables)

2. Simplex Method (2 or more than two variables)

   i) simple case (only for slack variables)

   ii) Big - M method

   Two phase method

   (Surplus variables are subtracted) & Artificial variables are added.

1. Graphical method (only 2 variables) :

Feasible Region :

The region that contains all passibly feasibly solution to the problem.

ie, those solution which satisfies all constraints af the problem

Questions :

1. The company making cold drinks has two Bottling plants in Towns T_1 and T_2 each plant produces Tea drinks A, B and C and their production capacity per day is given below

cold drinks	plant at	
	T_1	T_2
A	6000	2000
B	1000	2500
C	3000	3000

The marketing department of the company forecasts a demand of 80,000 (A), 22,000 (B), 40,000 (C) during June. The operating cost per day of plants T_1 and T_2 are Rs. 6000 and Rs. 4000 respectively. Find graphically the no. of. days for which each plant must be run in June so as to minimize operating cost by meeting the market demand.

sol: complete form of LPP is

$$\text{Min } z = 6000x_1 + 4000x_2$$

Subject to constraints

$$6000x_1 + 2000x_2 \geq 80{,}000 \quad (A)$$

$$1000x_1 + 2500x_2 \geq 22{,}000 \quad (B)$$

$$3000x_1 + 3000x_2 \geq 40{,}000 \quad (C)$$

where $x_1, x_2 \geq 0$

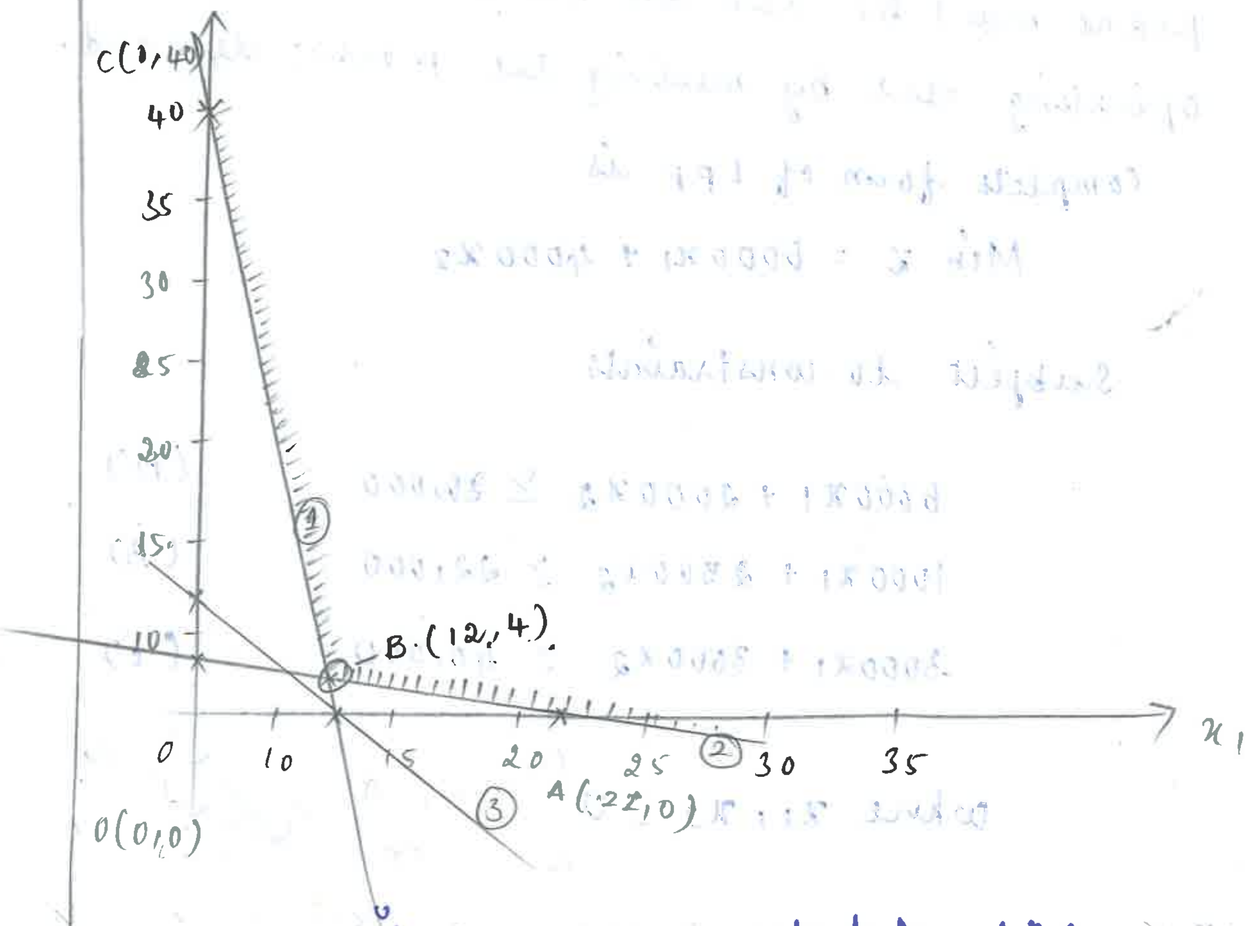
÷2000 $3x_1 + x_2 \geq 40$ → ①

÷1000 $x_1 + 2.5x_2 \geq 22$ → ②

÷1000 $3x_1 + 3x_2 \geq 40$ → ③

By graphical Method.

~~Vertices~~	~~objective $Min z = 6000x_1 + 4000x_2$~~
~~1. Let $3x_1 + x_2 = 40$ $(0,40)$ & $(40/3, 0)$ 13.33~~	~~At A $(0, 8.6)$ $Min z = 6000(0) + 4000(8.6)$ $= 35,200$~~
~~2. Let $x_1 + 2.5x_2 = 22$ $(0, \frac{22}{25}), (22, 0)$ 8.8~~	~~At B $(7.56, 5.77)$ =~~
~~3. Let $3x_1 + 3x_2 = 40$ $(0, \frac{40}{3})$ & $(\frac{40}{3}, 0)$ 13.3 13.3~~	~~At C $(13.33, 0)$ =~~



Feasible region bounded by ABC.

~~② & ③~~

~~$x_1 + 2.5x_2 = 22$~~

~~$3x_1 + 3x_2 = 40$~~

~~× 3 in ②~~

~~$\cancel{3x_1} + 7.5x_2 = 66$~~

~~$\cancel{3x_1} + 3x_2 = 40$~~

~~$4.5x_2 = 26$~~

~~$x_2 = \frac{26}{4.5}$~~

~~$\boxed{x_2 = 5.77}$~~

~~x_2 in ②~~

~~$x_1 + 2.5x_2 = 22$~~

~~$x_1 + 2.5(5.77) = 22$~~

~~$\boxed{x_1 = 7.56}$~~

solving ① & ②

$\cancel{3x_1} + x_2 = 40$

× 3 $\cancel{3x_1} + 7.5x_2 = 66$

$\cancel{-}6.5x_2 = \cancel{-}26$

$\boxed{x_2 = 4}$

x_2 in ①

$3x_1 + 4 = 40$

$3x_1 = 36$

$\boxed{x_1 = 12}$

pts	obj : Min z = $6000x_1 + 4000x_2$
A(22,0)	1,3,2000
B(12,4)	88,000
C(0,40)	1,60,000

∴ Min z = 88,000 at B(12,4).

2. Use the graphical method to solve the entity

$\text{Max } z = 3x_1 + 2x_2$, sub to $2x_1 + x_2 \le 40$

$x_1 + x_2 \le 24$

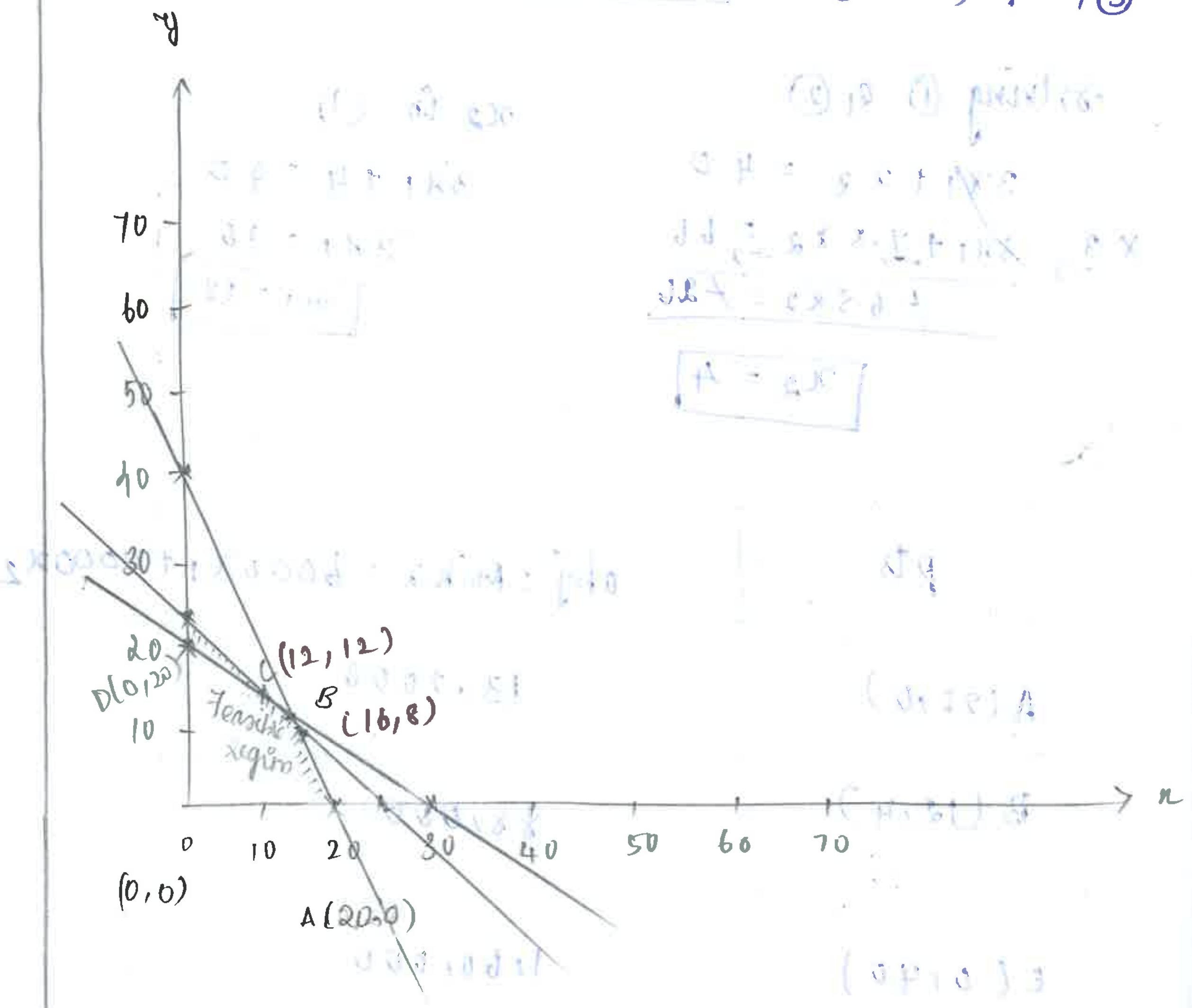
$2x_1 + 3x_2 \le 60$ and $x_1, x_2 \ge$

<u>Soln</u> :

Let $2x_1 + x_2 = 40 \Rightarrow (0, 40)\ (20, 0) \rightarrow ①$

$x_1 + x_2 = 24 \Rightarrow (0, 24)\ (24, 0) \rightarrow ②$

$2x_1 + 3x_2 = 60 \Rightarrow (0, 20)\ (30, 0) \rightarrow ③$



solving ① & ②

$$2x_1 + \cancel{x_2} = 40$$
$$(-)\quad \underset{(-)}{x_1} + \cancel{x_2} = \underset{(-)}{24}$$
$$\boxed{x_1 = 16} \rightarrow \text{substitute in ②}$$

$$x_1 + x_2 = 24$$
$$16 + x_2 = 24$$
$$\boxed{x_2 = 8}$$

∴ B(16, 8)

② & ③

$$\cancel{2x_1} + 2x_2 = 48$$
$$(-)\quad \cancel{2x_1} + \underset{(-)}{3x_2} = \underset{(-)}{60}$$
$$-x_2 = -12$$
$$\boxed{x = 12}$$

in ②

$$x_1 + x_2 = 24$$
$$12 + x_2 = 24$$
$$\boxed{x_2 = 12}$$

∴ C(12, 12)

The feasible region is bounded by OABCD.

points	obj : Max $z = 3x_1 + 2x_2$
O(0,0)	0
A(20,0)	Max z = 60
B(16,8)	Max z) = 64
C(12,12)	Max z = 60
D(0,20)	Max z = 40.

∴ Max z = 64 at (16, 8)

3. Use the graphical Method to solve the following LPP

$\text{Min } z = 20x_1 + 10x_2$ sub to $x_1 + 2x_2 \leq 40$

$3x_1 + x_2 \geq 30$

$4x + 3x_2 \geq 60$

and $x_1, x_2 \geq 0$

Soln :

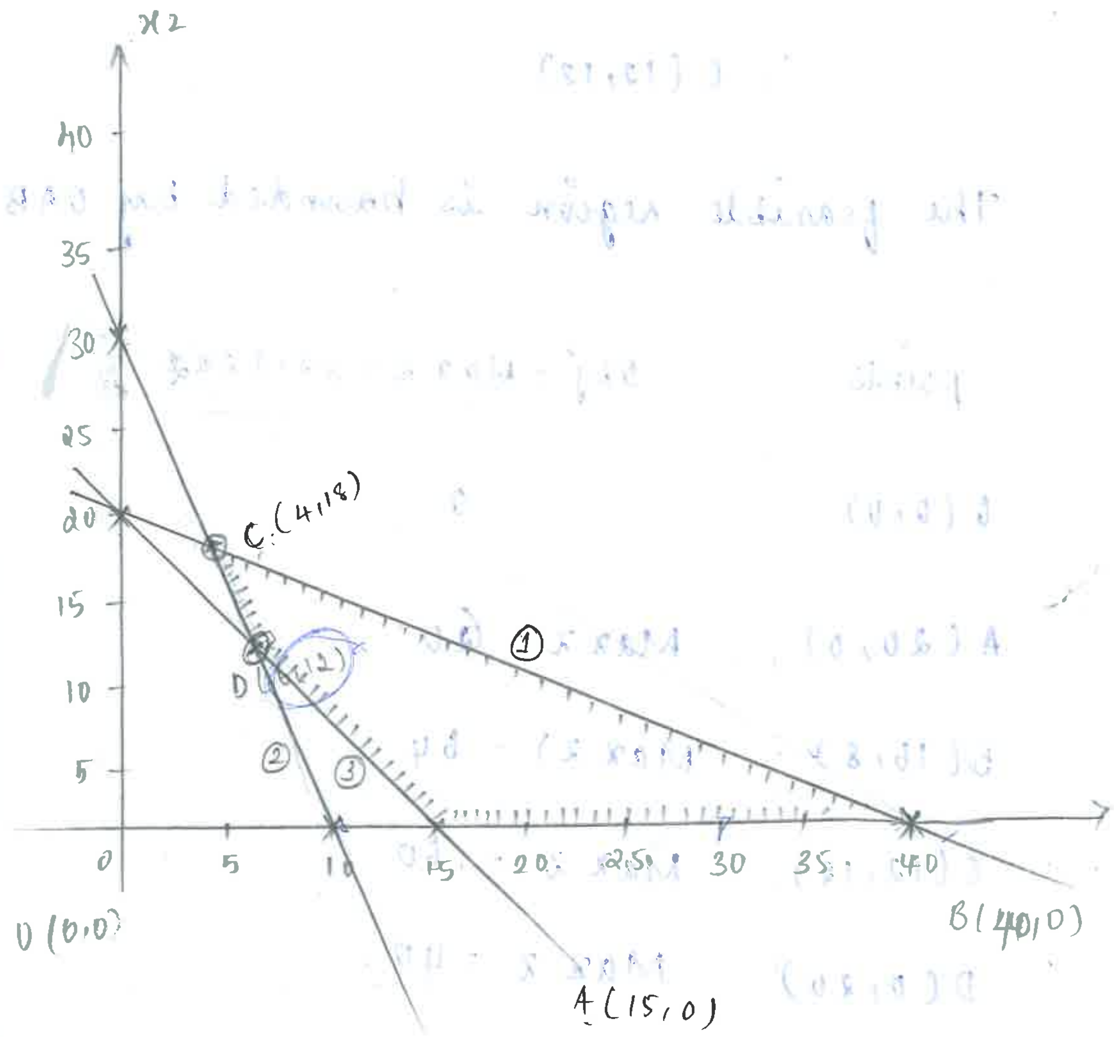
Let $x_1 + 2x_2 = 40 \Rightarrow (0, 20), (40, 0) \Rightarrow$ ①

$3x_1 + x_2 = 30 \Rightarrow (0, 30), (10, 0) \Rightarrow$ ②

$4x_1 + 3x_2 = 60 \Rightarrow (0, 20), (15, 0) \Rightarrow$ ③

$9x_1 + x_2 = 30$

graphical method :



solving ① & ②

$x_1 + 2x_2 = 40$

(×2) $6x_1 + \cancel{2x_2} = 60$
$\;(-)\quad(-)\quad(-)$

$\cancel{-}5x_1 = \cancel{-}20$

$\boxed{x_1 = 4}$

in ① $4 + 2x_2 = 40$

$2x_2 = 36$

$\boxed{x_2 = 18}$

∴ C(4,18)

solving ② & ③

(×3) $9x_1 + \cancel{3x_2} = 90$

$(-)\; 4x_1 + \cancel{3x_2} = 60$
$\quad(-)\quad(-)$

$5x_1 = 30$

$\boxed{x_1 = 6}$

in ② $3x_1 + x_2 = 30$

$18 + x_2 = 30$

$\boxed{x_2 = 12}$

∴ D(6,12)

The feasible region is bounded by ABCD.

points	obj = Min z = $20x_1 + 10x_2$
A(15,0)	300
B(40,0)	800
C(4,18)	260 ✓
D(6,12)	240 Min (z)

∴ Min z = 240 at D(6,12)

Type 2. Simplex Method.

1. Use simplex method to solve the LPP.

$\text{Max } z = 4x_1 + 10x_2$ sub to $2x_1 + x_2 \le 50$

$2x_1 + 5x_2 \le 100$

$2x_1 + 3x_2 \le 90$

and $x_1, x_2 \ge 0$

Soln.:- standard form of the given LPP is

$$\text{Max } z = 4x_1 + 10x_2 + 0s_1 + 0s_2 + 0s_3$$

sub to

$$2x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 50$$

$$2x_1 + 5x_2 + 0s_1 + s_2 + 0s_3 = 100$$

$$2x_1 + 3x_2 + 0s_1 + 0s_2 + s_3 = 90$$

and $x_1, x_2, s_1, s_2, s_3 \ge 0$.

		c_j		4	10	0	0	0	
	C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3	Y_B/col θ
	0	s_1	50	2	1	1	0	0	$50/1$
	0	s_2	$\boxed{100}$	2	⑤ pivot element	0	1	0	$\frac{100}{5}$
	0	s_3	90	2	3	0	0	1	$\frac{90}{3}$
		$z_j - c_j$	0	-4	-10 ↑ entering column most (-ve)	0	0	0	
$S_1 - x_2$	0	s_1	30	$8/5$	0	1	$-1/5$	0	
÷5 to make pivot 1	10	x_2	20	$2/5$	1	0	$1/5$	0	
$S_3 - 3x_2$	0	s_3	30	$4/5$	0	0	$-3/5$	1	
		$z_j - c_j$	200	0	0	0	2	0	

∴ all $z_j - c_j \geq 0$, we obtained the optimal solution.

∴ The optimal solution is $\boxed{\text{Max } Z = 200; \text{ at } x_1 = 0;\ x_2 = 20}$.

Simplex Algorithm : [Maximisation] .

Step 1 : Linear program is converted into a standard form (using slack (+s) and surplus variables) (-s)

Step 2 : All RHS values of the constraints are verified are in non - negativity (otherwise multiply by -1) .

Step 3 : The inequality of the constraints are converted into inequality as slack and surplus variables .

Step 4 : An initial solution to the problem is obtained and it entered in the first column.

leaving row (least +ve)

Step 5 : Find $z_j - c_j = c_B X_j - c_j$

[If $z_j - c_j \geq 0$, optimal solution is obtained]

Step 6 : Find X_B/which in entering , row corresponds $\min^m$ ratios is known as leaving row . The intersection of entering column and leaving row is known as Pivot (Key) element.

step 7 : Pivot element convert to unity, all others elements are divided by using elementary row to convert a zeros.

Step 8 : step 5 is repeated till the optima solution is obtained.

Remark :

$$\boxed{\text{Min } z = -\text{Max } (-z)}$$

2. Solve the following LPP for non-negativity of $x_1, x_2,$ which

Internal 1

$$\text{Max } z = 3x_1 + 2x_2 + 5x_3$$

sub to $x_1 + 4x_2 \le 420$

$3x_1 + 2x_3 \le 460$

$x_1 + 2x_2 + x_3 \le 430$ and $x_1, x_2, x_3 \ge$

soln :- standard form of the given LPP is

$$\text{Max } z = 3x_1 + 2x_2 + 5x_3 + 0s_1 + 0s_2 + 0s_3$$

Sub to

$$x_1 + 4x_2 + 0x_3 + 0s_2 + s_1 + 0s_3 = 420$$

$$3x_1 + 2x_3 + s_2 + 0s_1 + 0s_3 = 460$$

$$x_1 + 2x_2 + x_3 + 0s_1 + 0s_2 + s_3 = 430$$

and $x_1, x_2, x_3 \ge 0$

s_1, s_2, s_3

c_j = 3 2 5 0 0 0 19 × B/column

C_B	y_B	x_B	x_1	x_2	x_3	s_1	s_2	s_3	θ
0	s_1	420	1	4	0	1	0	0	-
0	s_2	460	3	0	$\boxed{2}$	0	1	0	230 →
0	s_3	430	1	2	1	0	0	1	430
$z_j - c_j$	0	-3	-2	-5 ↑	0	0	0		θ
0	s_1	420	1	4	0	1	0	0	105
5	x_3	230	3/2	0	1	0	1/2	0	-
0	s_3	200	-1/2	$\boxed{2}$	0	0	-1/2	1	100 →
$z_j - c_j$	1150	9/2	-2 ↑	0	0	5/2	0		
0	s_1	20	0	0	0	1	0	-2	
5	x_3	230	3/2	0		0	1/2	0	
2	x_2	100	-1/4	1	0	0	-1/4	1/2	
$z_j - c_j$	1350	4	0	0	0	2	1		

pivot element

÷2

$S_3 - x_3$

$S_1 - 4x_2$

÷2

$S_1 - 4x_2$

∴ all $z_j - c_j \geq 0$, we obtain the optimal solution.

The optimal solution is

$\boxed{\text{Max } z = 1350 \text{ ; at } x_1 = 0, x_2 = 100, x_3 = 230}$

Max z = $3x_1 + 2x_2 + 5x_3$

= 0 + 2 × 100 + 5 × 230

= 200 + 1150

= 1350

3. Solve the following LPP using Simplex Method.

Inter-1

$$\text{Min } z = x_2 - 3x_3 + 2x_5$$

sub to

$$3x_2 - x_3 + 2x_5 \le 7$$

$$-2x_2 + 4x_3 \le 12$$

$$-4x_2 + 3x_3 + 8x_5 \le 10$$

and $x_2, x_3, x_5 \ge 0$.

Soln:

The standard form of LPP is.,

$$\text{Max } z = -x_2 + 3x_3 - 2x_5 \quad [\therefore \text{Min } z = -\text{Max}(-$$

sub to

$$3x_2 - x_3 + 2x_5 \overset{+s_1}{\le} 7$$

$$-2x_2 + 4x_3 \overset{0s_1 + 0x_5 +}{+} + s_2 + 0s_3 = 12$$

$$-4x_2 + 3x_3 + 8x_5 + 0s_1 + 0s_2 + 0s_3 = 10$$

and $x_2, x_3, x_5 \ge 0$.
s_1, s_2, s_3

$-\frac{3}{2} \times 1$

			c_j =	-1	3	-2	0	0	0	
	CB	YB	XB	x_2	x_3	x_5	s_1	s_2	s_3	θ
	0	s_1	7	3	-1	2	1	0	0	-7
	0	s_2	12	-2	4	0	0	1	0	3
	0	s_3	10	-4	3	8	0	0	1	3.3
$-4+\frac{1}{2}$ $-\frac{8+1}{2} \cdot \frac{1}{2}$		$z_j - c_j$	0	1	-3 ↑ entering column	+2	0	0	0	
$s_1 + x_3$	0	s_1	10	5/2	0	2	1	1/4	0	4 ←
÷4	3	x_3	3	-1/2	1	0	0	1/4	0	-
$s_3 - 3x_3$	0	s_2	1	-7/2	0	8	0	-3/4	1	-

	$z_j - c_j$		9	$-1/2$	0	2	0	$3/4$	0	
$\div 5/2$	-1	x_2	4	1	0	$4/5$	$2/5$	[illegible]	0	
$\frac{1}{2}x_2 + x_3$	3	x_3	5	0	1	$2/5$	$1/5$	$3/10$	0	
$x_2 + s_3$	0	s_3	11	0	0	10	1	$-1/2$	1	
	$z_j - c_j$		11	0	0	$6/5$	$1/5$	$4/5$	0	

Since all $z_j - c_j \geq 0$, we obtained the optimal solution.

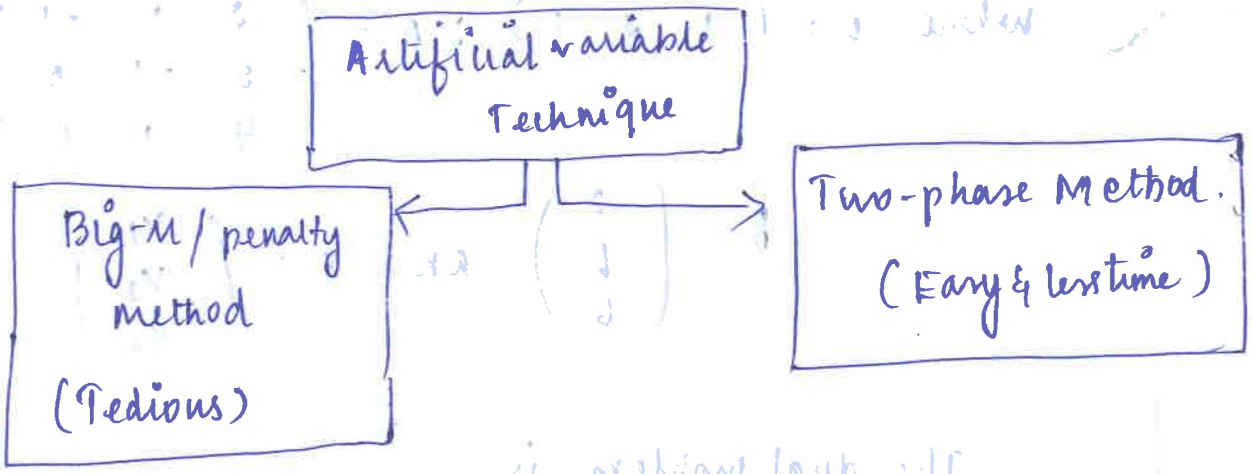
$\therefore$ The optimal solution is Max $z = 11$ at $x_2 = 4$;
$x_3 = 5$;
$x_5 = 0$.

$\therefore$ Min $z = -$ Max $(-z) = -11$

$\therefore$ $\boxed{\text{Min } z = -11 \text{ at } x_2 = 4;\ x_3 = 5;\ x_5 = 0}$

Artificial variable technique in simplex Method :-

when the constraints with '$\geq$' we can use either Big-M or Two phase method.



Duality :

Each LPP stated in its original form (primal) gets associated with another LPP (dual). This is known as Duality in LPP.

1. Write the standard form of the following primal LPP

$$\text{Max } z = x_1 + 2x_2 + x_3$$

sub to $2x_1 + x_2 - x_3 \le 2$

$-2x_1 + x_2 - 5x_3 \ge -6$

$4x_1 + x_2 + x_3 \le 6$ and $x_1, x_2, x_3 \ge 0$

soln: $\text{Max } z = x_1 + 2x_2 + x_3$

The primal LPP is

A b

$2x_1 + x_2 - x_3 \le 2$

$2x_1 - x_2 + 5x_3 \le 6$

$4x_1 + x_2 + x_3 \le 6$.

$\therefore \text{Max } z = c$

sub to

$Ax \le b$

and $x \ge 0$

where $c = (1 \quad 2 \quad 1)$; $A = \begin{pmatrix} 2 & 1 & -1 \\ 2 & -1 & 5 \\ 4 & 1 & 1 \end{pmatrix}$

$b = \begin{pmatrix} 2 \\ 6 \\ 6 \end{pmatrix}$ and $X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$.

The dual problem is

$\text{Min } W = b^T y$

sub to $A^T y \ge c^T$

and $y \ge 0$

The dual LPP form is

$$\text{Min } W = \begin{pmatrix} 2 & 6 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

sub to $\begin{pmatrix} 2 & 2 & 4 \\ 1 & -1 & 1 \\ -1 & 5 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $y_1, y_2, y_3 \geq 0$

ie... Min $W = 2y_1 + 6y_2 + 6y_3$

sub to

$2y_1 + 2y_2 + 4y_3 \geq 1$

$y_1 - y_2 + y_3 \geq 2$

$-y_1 + 5y_2 + y_3 \geq 1$ and $y_1, y_2, y_3 \geq 0$.

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2. Find the dual of the LPP

Max $x = 3x_1 - x_2 + x_3$

sub to

$4x_1 - x_2 \leq 8$

$8x_1 + x_2 + 3x_3 \geq 12$

$5x_1 - 6x_3 \leq 13$ and $x_1, x_2, x_3 \geq 0$

The primal LPP is

Max $x = 3x_1 - x_2 + x_3$

sub to

$4x_1 - x_2 \leq 8$

$-8x_1 - x_2 - 3x_3 \leq -12$

$5x_1 - 6x_3 \leq 13$

where $c = (3 \quad -1 \quad 1)$; $A = \begin{pmatrix} 4 & -1 & - \\ -8 & -1 & - \\ 5 & 0 & - \end{pmatrix}$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} ; \quad b = \begin{pmatrix} 8 \\ -12 \\ 13 \end{pmatrix}$$

The dual problem is

$\text{Min } W = B^T y$

sub to

$A^T y \geq c^T$

and $y \geq 0$

$$\text{Min } W = (8 \quad -12 \quad 13) \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

sub to

$$\begin{pmatrix} 4 & -8 & 5 \\ -1 & -1 & 0 \\ 0 & -3 & -6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \geq \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

The dual form

$\text{Min } w = 8y_1 - 12y_2 + 13y_3$

sub to

$4y_1 - 8y_2 + 5y_3 \geq 3$

$-y_1 - y_2 \geq -1$

$-3y_2 - 6y_3 \geq 1$

and $y_1, y_2, y_3 \geq 0$

c = obj
A = Const
b = Cons Coeff

8. Write the dual of the LPP.

$$\text{Min } z = 3x_1 - 2x_2 + 4x_3$$

Sub to

$$3x_1 + 5x_2 + 4x_3 \geq 7$$
$$6x_1 + x_2 + 3x_3 \geq 4$$
$$7x_1 - 2x_2 - x_3 \leq 10$$
$$x_1 - 2x_2 + 5x_3 \geq 3$$
$$4x_1 + 7x_2 - 2x_3 \geq 2.$$

Soln:

Min z ↓ $3x_1 - 2x_2 + 4x_3$ — sub to ↳

$$-3x_1 - 5x_2 - 4x_3 \leq -7$$
$$-6x_1 - x_2 - 3x_3 \leq -4$$
$$7x_1 - 2x_2 - x_3 \leq 10.$$
$$-x_1 + 2x_2 - 5x_3 \leq -3$$
$$-4x_1 - 7x_2 - 2x_3 \leq -2.$$

The dual problem is

$$\text{Max } W = b^T y$$

$$\text{Sub to } A^T y \geq c^T \text{ and } y \geq 0.$$

ie.
$$\text{Max } W = \begin{pmatrix} -7 & -4 & 10 & -3 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix}$$

Sub to

$$\begin{pmatrix} -3 & -6 & 7 & -1 & -4 \\ -5 & -1 & -2 & 2 & -7 \\ -4 & -3 & -1 & -5 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} \geq \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$$

ie. $\text{max } W = (+7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5)$
× (-)

sub to $3y_1+6y_2-7y_3+y_4+4y_5 \le -3$

X(-) $5y_1+y_2+2y_3-2y_4+7y_5 \le 2$

$4y_1+3y_2+y_3+5y_4-2y_5 \le -4$

and $y_1, y_2, y_3, y_4, y_5 \ge 0$.

Sensitivity Analysis :

The investigations that deal with changes with optimal solutions due to distinct variations with the parameters. a_{ij}, b_i and c_j are called sensitivity analysis or post optimality analysis.

1. Enter-1 Graphical

An Automobile manufacturer makes automobile and trucks in a factory that is divided into 2 shop. Shop A which performs the basic assembly operation must work 5 man-days on each truck. but only 2 man-days on each automobiles. shop B which performs finishing operation must work. 3 man-days an each truck or automobile. That it produces because of men and machine limitations shop A has 180 man days per week available while shop B has 135 man days per week. the manufacturer makes a profit of Rs. ₹ 300 on each truck and Rs ₹ 200 each automobile How many each should be produce to maxim the profit?

The LPP is

$\text{Max } z = 300x_1 + 200x_2$

Sub to

$5x_1 + 2x_2 \leq 180$

$3x_1 + 3x_2 \leq 135$

and $x_1, x_2 \geq 0$.

∴ The std form of LPP is

$\text{Max } z = 300x_1 + 200x_2 + 0s_1 + 0s_2$

sub to

$5x_1 + 2x_2 + s_1 + 0s_2 = 180$

$3x_1 + 3x_2 + s_2 + 0s_1 = 135$ and $x_1, x_2, s_1, s_2 \geq 0$

		c_j		300	200	0	0	
	CB	YB	XB	x_1	x_2	s_1	s_2	θ
$\frac{180}{5}$ 36. $\frac{135}{3}$ 45	0	s_1	180	5 → pivot	2	1	0	36 ← (least +ve)
	0	s_2	135	3	3	0	1	45
	$z_j - c_j$		0	-300 ↑(most -ve)	-200	0	0	
$x_1 \div 5$	300	x_1	36	1	2/5	pivot 1/5	0	90
$s_2 - 3x_1$	0	s_2	27	0	9/5	-3/5	+1	15
	$z_j - c_j$		10800	0	-80 ↑(most -ve)	60	0	
$x_1 - 2/5 x_2$	300	x_1	30	1	0	1/3	-2/9	
	200	x_2	15	0	1	-1/3	5/9	
$\div 9/5$	$z_j - c_j$		12000	0	0	33.33	44.44	

$\therefore z_j - c_j \geq 0$, we obtained the optimal solution.

The optimal solution is $\boxed{\text{Max } Z = 12000}$

at $x_1 = 30$, $x_2 = 15$.

---

2. Using simplex method, solve the LPP

$$\text{Max } z = 3x_1 - x_2$$

sub to

$$x_1 - x_2 \leq 10$$
$$x_1 \leq 20$$

and $x_1, x_2 \geq 0$.

Soln: The standard form of LPP is

$$\text{Max } z = 3x_1 - x_2 + 0s_1 + 0s_2$$

sub to

$$x_1 - x_2 + s_1 + 0s_2 = 10$$
$$x_1 + 0x_2 + 0s_1 + s_2 = 20$$

C_j 3 -1 0 0

C_B	Y_B	X_B	x_1	x_2	S_1	S_2	θ
0	S_1	10	1 pivot	-1	1	0	10 ← least +ve
0	S_2	20	1	0	0	1	20
$z_j - c_j$		0	-3 ↑ entering column	1	0	0	
3	x_1	10	1	-1	1	0	-10
0	S_2	10	0	1	-1	1	10 ← least +ve
$z_j - c_j$		30	0	-2 ↑	3	0	

$S_2 - x_1$

x_1+x_2

3	x_1	20	1	0	0	1
-1	x_2	10	0	1	-1	1
z_j-y_i		50	0	0	1	2

$\therefore$ all $z_j - y_i \geq 0$, the optimal solution is obtained.

The optimal solution is Max $z = 50$

$\boxed{x_1 = 20, x_2 = 10}$

27|08|22

Unit - 2

TRANSPORTATION AND NETWORK MODELS

Assignment models :

A problem in which 'n' different facilities are assigned to 'n' different tasks such a problem is known as assignment problem

2m Applications :

* Assignment of operations to job

* Machines allocation for optimal space utilization .

2m Assignment algorithm (Hungarian Method) :

Step 1 : Row reduction : Lowest cost of the Balanced matrix in the row is subtracted from each row.

Step 2 : Column Reduction : Lowest cost in the column is subtracted from each column of these new cost matrix.

Step 3 : Minimum no. of lines are drawn to cover all zeros.

order of matrix ↑

Step 4 : If the minimum lines are less than n for n x n matrix, then choose the least number does not covered by lines .

step 5 : This number subtracted from all elements that are not covered by lin and added at the paint of Intersection.

step 6 : A row is located which contains only one zero element, the job corresponding to this element i assigned to person, all zeros in the column or row corresponding t the element are crossed out.

If any of the row or column is not assigned then repeat step 5 by choosing other least elemen which is not covered by the lines.

1. Find the optimal solution of the following assignment problem using Hungarian Metho

$$
\begin{array}{ccc}
 & & \text{Job} \\
 & & \begin{array}{cccc} A & B & C & D \end{array} \\
\text{Worker} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 45 & 40 & 51 & 67 \\ 57 & \textcircled{42} & 63 & 55 \\ 49 & 52 & 48 & 64 \\ 41 & 45 & 60 & 55 \end{pmatrix}
\end{array}
$$

Soln: ∴ The order of the matrix is 4 (i.e, 4×4), i.e, it is a balanced matrix

1. Row reduction :

$$\begin{pmatrix} 5 & 0 & 11 & 27 \\ 15 & 0 & 21 & 13 \\ 1 & 4 & 0 & 16 \\ 0 & 4 & 19 & 14 \end{pmatrix}$$

2. Column Reduction :

$$\begin{pmatrix} 5 & 0 & 11 & 14 \\ 15 & 0 & 21 & 0 \\ 1 & 4 & 0 & 3 \\ 0 & 4 & 19 & 1 \end{pmatrix}$$

∵ No. of lines = 4 = order of the matrix

i.e, optimal solution is obtained .

$$\begin{array}{c|cccc} & A & B & C & D \\ \hline 1 & 5 & \boxed{0} & 11 & 14 \\ 2 & 15 & \cancel{0} & 21 & \boxed{0} \\ 3 & 1 & 4 & \boxed{0} & 3 \\ 4 & \boxed{0} & 4 & 19 & 1 \end{array}$$

Assignment :

1 → B

2 → D

3 → C

4 → A

∴ optimum cost (min cost) = ₹40 + 55 + 48 + 41

= ₹ 184.

2. Processing times for job when allocated to 5 machines are indicated below. Assigning the machine for the job shows the processing time is minimum.

$$\begin{array}{c} & M_1 & M_2 & M_3 & M_4 & M_5 \\ J_1 & 9 & 25 & 58 & 11 & 19 \\ J_2 & 43 & 78 & 72 & 50 & 63 \\ J_3 & 41 & 28 & 91 & 37 & 45 \\ J_4 & 74 & 42 & 27 & 49 & 39 \\ J_5 & 36 & 11 & 57 & 22 & 25 \end{array}$$

Determine the optimal assignment.

Soln :

∴ The order of the matrix is 5 ie $\boxed{(5 \times 5)}$,
ie, it is a balanced matrix.

1. Row Reduction :-

$$\begin{pmatrix} 0 & 16 & 49 & 2 & 10 \\ 0 & 35 & 29 & 7 & 20 \\ 13 & 0 & 63 & 9 & 17 \\ 47 & 15 & 0 & 22 & 12 \\ 25 & 0 & 11 & 46 & 14 \end{pmatrix}$$

2. column reduction :

$$\begin{pmatrix} 0 & 16 & 49 & 0 & 0 \\ 0 & 35 & 29 & 5 & 10 \\ 3 & 0 & 63 & 7 & 7 \\ 47 & 15 & 0 & 20 & \textcircled{2} \\ 25 & 0 & 0 & 40 & 4 \end{pmatrix}$$

∴ No. of lines = 4 ≠ order (5)

$$\begin{pmatrix} \textcircled{2} & \textcircled{18} & 51 & 0 & 0 \\ 0 & 35 & 29 & 3 & 8 \\ 13 & 0 & 63 & 5 & 5 \\ \textcircled{47} & \textcircled{15} & 0 & 18 & 0 \\ 25 & 0 & 7 & 40 & \textcircled{2} \end{pmatrix}$$

add min intersection

∴ No. of lines = 4 ≠ order (5)

$$\begin{pmatrix} 04 & 20 & 51 & 0 & 0 \\ 0 & 35 & 27 & 1 & 6 \\ 13 & 0 & 61 & 3 & 3 \\ 49 & 17 & 0 & 18 & 0 \\ 25 & 0 & 5 & 38 & 0 \end{pmatrix}$$

∴ No. of lines = 5 = order.

we obtained the optimal assignment.

$$\begin{array}{cccccc} & M_1 & M_2 & M_3 & M_4 & M_5 \\ J_1 & 4 & 20 & 51 & \boxed{0} & \cancel{0} \\ J_2 & \boxed{0} & 35 & 27 & 9 & 6 \\ J_3 & 13 & \boxed{0} & 61 & 3 & 3 \\ J_4 & 49 & 17 & \boxed{0} & 18 & \cancel{0} \\ J_5 & 25 & \cancel{0} & 38 & 5 & \boxed{0} \end{array}$$

$\therefore$ optimal schedule is

$J_1 \rightarrow M_4$ $\quad J_4 \rightarrow M_3$

$J_2 \rightarrow M_1$ $\quad J_5 \rightarrow M_5$

$J_3 \rightarrow M_2$

$\therefore$ optimal solution is $= 11 + 43 + 28 + 27 + 25$

$= 134$ hrs.

---

3. A company has four machines to do three jobs and each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table.

Machines

$$\text{jobs}\quad \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}$$

$$\begin{array}{ccccc} & 1 & 2 & 3 & 4 \\ A & 18 & 24 & 28 & 32 \\ B & 8 & 13 & 17 & 19 \\ C & 10 & 15 & 19 & 22 \end{array} \quad 3 \times 4 =$$

what are job assignments which will minimize the cost ?

Soln:

$$\begin{pmatrix} 18 & 24 & 28 & 32 \\ 8 & 13 & 17 & 19 \\ 10 & 15 & 19 & 22 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{4 \times 4}$$ [∴ introducing Dummy row]

i) Row Reduction

$$\begin{pmatrix} 0 & 6 & 10 & 14 \\ 0 & 5 & 9 & 11 \\ 0 & 5 & 9 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

ii) Column Reduction

same as Row Reduction.

∴ No. of lines ≠ order (4)

$$\begin{pmatrix} 0 & 1 & 5 & 9 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 4 & 7 \\ 5 & 0 & 0 & 0 \end{pmatrix}$$ No. of lines = 3 ≠ order (4)

$$\begin{pmatrix} 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 9 & 4 & 0 & 0 \end{pmatrix}$$

∴ No. of lines = 4 = order (4).

we obtained the optimal assignment

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline A & \boxed{0} & 1 & 1 & 5 \\ B & \cancel{0} & \boxed{0} & \cancel{0} & 2 \\ C & \cancel{0} & \cancel{0} & \boxed{0} & 3 \\ D & 9 & 4 & \cancel{0} & \boxed{0} \end{array}$$

∴ The optimal schedule is

A → 1　　C → 3

B → 2　　D → 4

∴ optimal solution is = 18 + 13 + 19 + 0
(min cost)　　= ₹ 50

Maximization problem :-

$$\boxed{\text{Max } z = -\min(-z)}$$

∴ choose largest element and subtract this b
rest, to convert it no minimization.

4. A company is faced with the problem of assigning 4 different sales man to 4 different teriton for promoting it sales. Territories are not equally rich in their sales patential and the Salesman also differ in their ability to promote sales. The following table gives expected annual sales (in thousand of Ru

for each sales man if assigned to various territories. Find the assignment of salesman so as to maximize the annual sales.

Territories

$$\begin{array}{cc|cccc} & & 1 & 2 & 3 & 4 \\ \text{Salesman} & 1 & \underline{60} & 50 & 40 & 30 \\ & & 40 & 30 & 20 & 15 \\ & 3 & 40 & 20 & 35 & 10 \\ & 4 & 30 & 30 & 25 & 20 \end{array}$$

4×4

To convert maximization as minimization subtract all elements by 60 (largest).

$$\begin{pmatrix} 0 & 10 & 20 & 30 \\ 20 & 30 & 40 & 45 \\ 20 & 40 & 25 & 50 \\ 30 & 30 & 35 & 40 \end{pmatrix}$$

Row Riduction :

2×3

$$\begin{pmatrix} 0 & 10 & 20 & 30 \\ 0 & 10 & 20 & 25 \\ 0 & 20 & 5 & 30 \\ 0 & 0 & 5 & 10 \end{pmatrix}$$

Column Reduction

$$\begin{pmatrix} 0 & 10 & 15 & 20 \\ 0 & 10 & 15 & 15 \\ 0 & 20 & 0 & 20 \\ \textcircled{0} & 0 & \textcircled{0} & 0 \end{pmatrix}$$

∴ No of lines = ③ ≠ order (4)

$$\begin{pmatrix} 0 & 0 & 15 & 10 \\ 0 & 0 & 15 & 5 \\ 0 & 10 & 0 & 10 \\ 10 & 0 & 10 & 0 \end{pmatrix}$$

No. of lines = 4 = order (4)

we obtained optimal solutn assignmen

$$\begin{array}{cc} & T \\ & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ S \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} \boxed{0} & \cancel{0} & 15 & 10 \\ \cancel{0} & \boxed{0} & 15 & 5 \\ \cancel{0} & 10 & \boxed{0} & 10 \\ 10 & \cancel{0} & 10 & \boxed{0} \end{pmatrix} \end{array}$$

∴ optimal schedule is

$S_1 \rightarrow T_1 \quad S_2 \rightarrow T_2 \quad S_3 \rightarrow T_3 \quad S_4 \rightarrow T_4$.

∴ (Max) optimal solution is = 60 + 30 + 35 + 20

= 145 { Thousands in Rs

= ₹1,45,000/-

Travelling salesman problem :- (salesman has t cover all the destinations and has to come bac to the same source.

1. Solve the following travelling salesman problem so as to minimize the cost per cycle

To city

$$\begin{array}{c|ccccc} \text{From city} & A & B & C & D & E \\ \hline A & - & 3 & 6 & \boxed{2} & 3 \\ B & 3 & - & 5 & 2 & 3 \\ C & 6 & 5 & - & 6 & 4 \\ D & \boxed{2} & 2 & 6 & - & 6 \\ E & 3 & 3 & 4 & 6 & - \end{array}$$

∴ 5×5

Soln: i) Row Reduction

$$\begin{pmatrix} - & 1 & 4 & 0 & 1 \\ 1 & - & 3 & 0 & 1 \\ 2 & 1 & - & 2 & 0 \\ 0 & 0 & 4 & - & 4 \\ 0 & 0 & 1 & 3 & - \end{pmatrix}$$

ii) column Reduction

$$\begin{pmatrix} - & 1 & 3 & 0 & 1 \\ 1 & - & 2 & 0 & 1 \\ 2 & 1 & - & 2 & 0 \\ 0 & 0 & 3 & - & 4 \\ 0 & 0 & 0 & 3 & - \end{pmatrix}$$

∴ No. of lines = 4 ≠ order (5)

$$\begin{pmatrix} - & 0 & 2 & 0 & 1 \\ 0 & - & 1 & 0 & 1 \\ 1 & 0 & - & 2 & 0 \\ 0 & 0 & 3 & - & 5 \\ 0 & 0 & 0 & 4 & - \end{pmatrix}$$

∴ No. of lines = 5 = order (5)

we obtained the optimal schedule.

$$\begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & - & \cancel{0} & 2 & \cancel{0} & \boxed{1} \\ B & \cancel{0} & - & 1 & \boxed{0} & 1 \\ C & 1 & \boxed{0} & - & 2 & \cancel{0} \\ D & \boxed{0} & \cancel{0} & 3 & - & 5 \\ E & \cancel{0} & \cancel{0} & \boxed{0} & 4 & - \end{array}$$

A → E

optimum schedule is

A → E, B → D, C → B; D → A; E → C.

A → E → C → B → D → A.

optimum value = 3 + 2 + 5 + 2 + 4

= 16.

2. Salesman estimates with the following would be co

a his route, visiting six cities are shown belo

To city

From city	1	2	3	4	5	6
1	∞	20	23	27	29	34
2	21	∞	29	32	31	24
3	25	28	∞	15	36	26
4	35	16	25	∞	23	18
5	23	40	23	31	∞	10
6	27	18	12	35	16	

Salesman can visit each of the city only once. Determine the optimum sequence, he should follow to minimize the total distance travelled.

The order of matrix is (6×6).

i) Row Reduction :-

$$\begin{pmatrix} \infty & 0 & 3 & 7 & 9 & 4 \\ \underline{2} & \infty & 0 & 7 & 12 & 5 \\ 10 & 13 & \infty & \underline{0} & 21 & 11 \\ 19 & \underline{0} & 9 & \infty & 7 & 2 \\ 13 & 30 & 13 & 21 & \infty & \underline{0} \\ 15 & 6 & \underline{0} & 23 & \underline{4} & \infty \end{pmatrix} 6 \times 6.$$

ii) column Reduction :

$$\begin{pmatrix} \infty & 0 & 3 & 7 & 5 & 14 \\ 0 & \infty & 0 & 7 & 8 & 8 \\ 8 & 13 & \infty & 0 & 14 & 11 \\ 17 & 0 & 9 & \infty & 3 & 2 \\ 11 & 30 & 13 & 21 & \infty & 0 \\ 13 & 6 & 0 & 23 & 0 & \infty \end{pmatrix}$$

No. of lines = 5 ≠ order (6)

$$\begin{pmatrix} \infty & 0 & 0 & 7 & 2 & 14 \\ 0 & \infty & 0 & 10 & 8 & 8 \\ 5 & 13 & \infty & 0 & 14 & 11 \\ 14 & 0 & 6 & \infty & 0 & 2 \\ 8 & 30 & 10 & 21 & \infty & 0 \\ 13 & 9 & 0 & 26 & 0 & \infty \end{pmatrix}$$

∴ No of. lines = 6 = order.

we obtained the optimum cycle.

$$\begin{pmatrix} \infty & \cancel{0} & \cancel{0} & 7 & \boxed{2} & 14 \\ \boxed{0} & \infty & \cancel{0} & 10 & 8 & 8 \\ 5 & 13 & \infty & \boxed{0} & 14 & 11 \\ 14 & \boxed{0} & 6 & \infty & \cancel{0} & 2 \\ 8 & 30 & 10 & 21 & \infty & \boxed{0} \\ 13 & 9 & \boxed{0} & 26 & \cancel{0} & 8 \end{pmatrix}$$

The optimum schedule,

1 → 5, 2 → 1, 3 → 4, 4 → 2, 5 → 6, 6 → 3

1 → 5 → 6 → 3 → 4 → 2 → 1

∴ optimum cost = 29+21 + 15 + 16 + 10+12

= ₹103/-

Transportation problem :

It deals with transportation of single product form several origin in several destination

Origin :

It is a location from which shipments or dispatched.

Destination :

which is a location from which shipment are transported.

Unit transport cost : It is a cost of transporting unit of consignment.

Feasible solution :

Methods to find Initial basic feasible solution (IBFS)

There are 3 methods :

1. North - west corner Rule (N-W corner Rule)
2. Matrix minimum method (or) least cost method (L
3. Vogel's approximation method (VAM).

(8m)

1. Solve the following TP. using N-W corner rule.

		1	2	3	4	supply
	A	40	25	22	23	100 −40 =60
origin	B	44	35	30	30	30
	C	38	38	28	30	20
Demand		40	20	60	30	150

soln:

∴ Tot supply = 150 = Tot Demand

By N-W corner rule

				Demand
$\boxed{40}$ ~~40~~	$\boxed{20}$ ~~25~~	$\boxed{40}$ 22	~~23~~	~~100~~ ~~60~~ ~~40~~
~~44~~	~~35~~	$\boxed{20}$ ~~30~~	$\boxed{10}$ ~~30~~	~~30~~ ~~10~~
~~38~~	~~38~~	~~28~~	$\boxed{20}$ 30	20
~~40~~	~~20~~	~~60~~ 20	~~30~~ 20	150

Supply

TP cost = 40×40 + 20×25 + 40×22 + 20×30 + 10×30 + 20×30 = ₹4480/-

2. Find the IBFS for the following. TP using least cost method (LCM) [MMM]

	R₁	R₂	R₃	R₄	supply
P₁	3	5	7	6	50
P₂	2	5	8	2	75
P₃	3	6	9	2	25
Demand	20	20	50	60	

soln:

					supply
	$\boxed{5}$ 3	$\boxed{20}$ 5	$\boxed{25}$ 7	6	~~50~~ ~~45~~ ~~25~~
	$\boxed{15}$ 2	5	8	$\boxed{60}$ 2	~~75~~ ~~15~~
	3	6	$\boxed{25}$ 9	2	25
Demand	~~20~~ 5	20	~~50~~ 25	60	150

$$TP\ cost = 5\times3 + 20\times5 + 25\times7 + 15\times2 + 60\times2 + 25\times9$$

$$= ₹665/-$$

---

4. Solve the following TP using N-W corner rule

	Destination				Supply
	1	2	3	4	11
I	21	16	25	13	13
II	17	18	14	23	19
III	32	27	18	41	$\boxed{43}$
Demand	6	10	12	15	

Total supply = 43 = Total Demand.

By N-W corner Rule,

$\boxed{6}$ 21	$\boxed{5}$ 16	25	13	~~11~~ 5
17	$\boxed{5}$ 18	$\boxed{8}$ 14	23	~~13~~ 8
32	27	$\boxed{4}$ 18	$\boxed{15}$ 41	~~19~~ 15
6	~~10~~ 5	~~12~~ 4	15	$\boxed{43}$

TP cost = 6 x 21 + 5 x 16 + 5 x 18 + 8 x 14 + 15 x 41 + 4 x 18

= ₹1095/-

---

3. Vogel's Approximation method (VAM):

1. Solve the following TP by VAM:

				Supply
10	20	5	7	10
13	9	12	8	20
4	5	7	9	30
14	7	4	0	40
3	12	5	19	50
Demand 60	60	20	10	$\boxed{150}$

Soln : Total Demand = 150 = Total supply .

least no two diffuence

	$\boxed{10}$ 10	20	5	7	10	(2)	(5)	(10) ←
	13	$\boxed{20}$ 9	12	8	20	(1)	(3)	(4)
	4	$\boxed{30}$ 5	7	9	30	(1)	(1)	(1)
	14	$\boxed{10}$ 7	$\boxed{20}$ 1	$\boxed{10}$ 0	~~40~~ ~~30~~ 10	(1)	(6) ←	(7)
	$\boxed{50}$ 3	12	5	19	50	(2)	(9)	(9) ←
	~~60~~ 50	60	20	10				
column diffrence	(1)	(2)	(4)	(7)				
	(1)	(2)						

↑ most value diffunce

No. of . Allocation $= 7 \neq \underset{\text{no. of. rows}}{m} + \overset{\text{no. of. columns}}{n} - 1 . (8)$

∴ This transportation problem is degeneracy .

∴ TP cost = 10×10 + 20×9 + 30×5 + 10×7 + 20×1 + 50×3

= ₹ 670/-

2. Solve the following by VAM :

				Supply.
21	16	25	13	11
17	18	14	23	13
32	27	18	41	19
Demand 6	10	12	15	42

soln : Total Demand = 42 = Total supply.

21	16	25	$\boxed{11}$ 13	~~11~~	(3)	
$\boxed{6}$ 17	$\boxed{3}$ 18	14	$\boxed{4}$ 23	~~13~~ ~~9~~ ^{3}	(5)	(3)
32	$\boxed{7}$ 27	$\boxed{12}$ 18	41	19	(9)	(9)
6	~~10~~ 7	12	~~15~~ 4			
(4)	(2)	(4)	(10)			
(15)	(9)	(4)	(18)			

No. of Allocation = 6 = m + n - 1 (6)

∴ This TP problem is not degeneracy.

TP cost = 6 x 17 + 3 x 18 + 7 x 27 + 12 x 18 + 4 x 2
11 x 13

= ₹ 796/-

Non-degenracy :

If the no. of allocations are equal to $m+n-1$, then the solution is non-degenracy.

degenracy :

If the no. of Allocation are less than $m+n-1$, then the solution is degenracy.

⑯ Test for optimality (Modified distribution method (or) MODI)

step 1 : Find IBFS by NW corner Rule or LCM or VAM method.

2 : check for non-degenracy.

i.e.. No. of Allocation $= m+n-1$

otherwise, introduce 'ϵ' in suitable independent position ($\epsilon \to 0$)

3 : For occupied or allocated cells, the relation which means

$c_{ij} = u_i + v_j$ has to be satisfied by assuming $(u_i = 0)$ or $(v_j = 0)$.

From maximum no. of allocations.

4 : Find $u_i + v_j$ for unallocated cells and find $d_{ij} = c_{ij} - (u_i + v_j)$.

If all $d_{ij} \geq 0$, then optimal solution obtained.

otherwise, if any of them is negative draw a closed path [horizontal or vertical] through allocated cells.

closed loop indicates [+θ, -θ]

alternatively at the corners then min (-θ) from allocated cells added with +θ and subtract with -θ.

5 : Repeat step 2 to 4 to test the optimality until all $d_{ij} \geq 0$.

1. Solve the TP when the unit transport costs, demand and supplies are as given below

	D_1	D_2	D_3	D_4	Supply
O_1	6	1	9	3	70
O_2	11	5	2	8	55
O_3	10	12	4	7	70
Demand	85	35	50	45	195 215

Soln :- Total Demand ≠ Total supply.

6	1	9	3	70
11	5	2	8	55
10	12	4	7	70
0	0	0	0	20
85	85	50	45	$\boxed{215}$

Total Dem
= Total
supply.

i) To find IBFS by VAM

				Supply
[65] 6	[5] 1	9	3	~~70~~ 5 (2) (2)
11	[30] 5	[25] 2	8	~~55~~ 25 (3) (3) (6)
10	12	[25] 4	[45] 7	70 (3) (3) (3)
[20] 0	0	0	0	20 (0)
~~85~~ 65	~~35~~ 30	~~50~~ 25	45	$\boxed{215}$
(6)	(1)	(2)	(3)	
(4)	(1)	(2)	(4)	
	(7)	(2)	(1)	

No. of. allocation = 7 = m + n − 1 = 7.

∴ The given problem is non-degenracy.

∴ TP cost = 65 × 6 + 5 × 1 + 30 × 5 + 25 × 2 + 25 × 4 + 45 × 7 + 20 × 0

= ₹ 1010.

ii) To find optimal soln (by MODI). Find $u_i + v_j$ [only for allocated cells]

					u_i
[65] 6 −θ	[5] 1 +θ	9	3	70	$u_1 = -4$
11	[30] 5	−θ [25] 2 +θ	8	55	$u_2 = 0$
10 −10	12	[25] 4 −θ	[45] 7	70	$u_3 = 2$
[20] 0	0	0	0	20	$u_4 = -10$

v_j $v_1 = 10$ $v_2 = 5$ $v_3 = 2$ $v_4 = 5$

For unallocated cells:

$d_{13} = 9 - (-4 + 2) \geq 0$

$c_{ij} - (u_i + v_j)$

$d_{14} = 3 - (-4+5) = 3 - 1 > 0$

$d_{21} = 11 - (10+0) = 1 > 0$

$d_{24} = 8 - (5+0) = 3 > 0$

$d_{31} = 10 - (2+10) = -2 < 0$

$\therefore \min(-\theta) = \min(25, 30, 65) = 25$

For allocated cells :

$\boxed{40}$ 6	$\boxed{30}$ 1	9	3	$u_1 = 0$
11	$\boxed{5}$ 5	$\boxed{50}$ 2	8	$u_2 = 4$
$\boxed{25}$ 10	12	4	7	$u_3 = 4$
0	0	0	0	$u_4 = -6$
$V_1 = 6$	$V_2 = 1$	$V_3 = -2$	$V_4 = 3$	

For unallocated cells :

$d_{13} = 9 - (0-2) = 11 > 0$

$d_{14} = 3 - (0+3) = 0$

$d_{21} = 11 - (4+6) > 0$

$d_{24} = 8 - (4+3) > 0$

$d_{32} = 12 - (4+1) > 0$

$d_{33} = 4 - (4-2) > 0$

$d_{42} = 0 - (-6+1) = 5 > 0$

$d_{43} = 0 - (-6-2) = 8 > 0$

$d_{44} = 0 - (-6+3) = 3 > 0$

$\therefore$ all $d_{ij} \geq 0$

the current solution is optimal solution

$\therefore$ optimal cost $= 40\times6 + 30\times1 + 5\times5 + 50\times2 + 25\times10 + 45\times7 + 20\times0$

$= ₹\,960/-$

2. Find the optimum solution by VAM,

					Supply
4	1	2	6	9	100
6	4	3	5	7	120
5	2	6	4	8	120
Demand 40	50	70	90	90	340

<u>soln</u>:- To obtain Feasible soln (by VAM)

$\boxed{30}$ 4	1	$\boxed{70}$ 2	6	9	~~100~~ 30	(1)	(2)←	(2)←	
6	4	3	$\boxed{30}$ 5	$\boxed{40}$ 7	120	(1)	(2)	(1)	(2)
$\boxed{10}$ 5	$\boxed{50}$ 2	6	$\boxed{60}$ 4	8	~~120~~ ~~20~~ 60	(2)	(1)	(1)	(4)↑
~~40~~ 10	50	70	~~90~~ 30	90	$\boxed{340}$				
(1)	(1)	(1)	(1)	(1)					
(1)			(1)	(1)					

No. of allocation $= 7 =$
$m+n-1$ $(3+\cancel{5}^{7}-1)$
TP is non degeneracy.

TP cost $= 30\times4 + 70\times2 + 30\times5 + 90\times7 + 10\times5 + 50\times2 + 60\times4 = ₹\,1430/-$

ii) To find optimal soln (By MODI method).

$C_{ij} = u_i + v_j$

[30] 4 $+\theta$	1	[70] 2 $-\theta$	6	9	$u_1 = 0$
6	4	3 $+\theta$	[30] 5 $-\theta$	[90] 7	$u_2 = 2$
[10] 5 $-\theta$	[50] 2	6	[61] 4 $+\theta$	8	$u_3 = 1$
$v_1 = 4$	$v_2 = 1$	$v_3 = 2$	$v_4 = 3$	$v_5 = 5$	

For unallocated cells.

$d_{12} = 1 - (0 + 1) = 0$

$d_{15} = 9 - (0 + 5) = 4$

$d_{14} = 6 - (0 + 3) = 3$

$d_{21} = 6 - (2 + 4) = 0$

$d_{22} = 4 - (2 + 1) = 1$

$d_{23} = 3 - (2 + 2) = -1 < 0$

$d_{33} = 6 - (2 + 1) = 3$

$d_{35} = 8 - (1 + 5) = 2.$

$\min(-\theta) = \min(70, 30, 10) = 10$

Allocated cells $c_{ij} = u_i + v_j.$

[40] 4	1 $+\theta$	[60] 2 $-\theta$	6	9	$u_1 = 0$
6	4	[10] 3 $+\theta$	[20] 5 $-\theta$	[90] 7	$u_2 = 1$
5	[50] 2 $-\theta$	6	[70] 4 $+\theta$	8	$u_3 = 0$
$v_1 = 4$	$v_2 = 2$	$v_3 = 2$	$v_4 = 4$	$v_5 = 6$	

$\boxed{40}$ 4	$\boxed{20}$ 1	$\boxed{40}$ 2	6	9
6	4	$\boxed{30}$ 3	5	$\boxed{90}$ 7
5	$\boxed{30}$ 2	6	$\boxed{80}$ 4	8

$V_1 = 4$ $V_2 = 1$ $V_3 = 2$ $V_4 = 3$ $V_5 = 6$

For unallocated cells.

$d_{14} = 6 - (0+3) = 3$

$d_{15} = 9 - (0+6) = 3$

$d_{21} = 6 - (5) = 1$

$d_{22} = 4 - (1+1) = 3$

$d_{24} = 5 - (4) = 1$

$d_{31} = 5 - (5) = 0$

$d_{33} = 6 - (3) = 3$

$\therefore$ all $d_{ij} \geq 0$, The current solution is optimal solution.

The optimal cost $= 40 \times 4 + 20 \times 1 + 40 \times 2 + 80 \times 3 + 90 \times 7 + 30 \times 2 + 80 \times 4$

= ₹ 1400/-.

Remark :

when no of allocation $< m+n-1$, then the tp is called degenracy.

To convert the degenracy to non-degenracy (No. of allocation $= m+n-1$).

By Introducing dummy variable $\epsilon \to 0$.

but it does not form loops.

TP on maximization

1. Solve the following TP to Maximize the profit

	40	25	22	33	100
	44	35	30	30	30
	38	38	28	30	70
Demand	40	20	60	30	

Since the MODI is maximization problem, we subtract largest value with remaining value.

4	19	22	11
0	9	14	14
6	6	16	14

$\boxed{10}$ 4	19	$\boxed{10}$ 22	$\boxed{30}$ 11	$\boxed{50}$ 0	100	(4) (4)
$\boxed{30}$ 0	9	14	14	0	30	(9) ←
6	$\boxed{20}$ 6	$\boxed{50}$ 16	14	0	~~70~~ 50	(6) (6)
~~40~~ 10	20	~~60~~ 10	30	50	$\boxed{200}$	
(4)	(3)	(2)	(3)	(0)		
(2)	(13)	(6)	(3)	(0)		

400
220
990
1320
780
1400

∴ No. of allocation = 7 = m+n−1 = 7.

∴ This is non-degeneracy.

TP cost = 10 × 40 + 10 × 22 + 30 × 33 × 50 × 0 + 30 × 44 + 20 × 38 + 50 × 98.

= ₹ 5090/-

To find optimal soln (By MODI method)

allocated cells ⇒ $c_{ij} = u_i + v_j$

$\boxed{10}$ 4 +θ	19	$\boxed{10}$ 22 −θ	$\boxed{30}$ 11	$\boxed{50}$ 0	$u_1 = 0$
$\boxed{30}$ 0 −θ	9	14 +θ	14	0	$u_2 = -4$
6	$\boxed{20}$ 6	$\boxed{50}$ 16	14	0	$u_3 = -6$
$v_1 = 4$	$v_2 = 12$	$v_3 = 22$	$v_4 = 11$	$v_5 = 50$	

−θ = −
+θ = +

For unallocated cells,

$d_{12} = 19 - (0 + 12) \geqslant 0$

$d_{22} = 9 - (12 - 4) > 0$

$d_{23} = 14 - (22 - 4) = 14 - 18 = -4.$

$\min(-\theta) = (10, 30) = 10$

$\boxed{20}$ 4	19	22	$\boxed{30}$ 11	$\boxed{50}$ 0	$u_1 = 0$
$\boxed{20}$ 0	9	$\boxed{10}$ 14	14	0	$u_2 = -4$
6	$\boxed{20}$ 6	$\boxed{30}$ 6	14	0	$u_3 = -2$
$v_1 = 4$	$v_2 = 8$	$v_3 = 18$	$v_4 = 11$	$v_5 = 0$	

allocated cells $c_{ij} = u_i +$

For unallocated cells,

$d_{12} = 19 - (8+0) > 0$

$d_{13} = 22 - (18+0) > 0$

$d_{22} = 9 - (8-4) > 0$

$d_{24} = 14 - (11-4) > 0$

$d_{25} = 0 - (0-4) > 0$

$d_{31} = 6 - (4-2) > 0$

$d_{34} = 14 - (11-2) > 0$

∴ all $d_{ij} \geq 0$, The current solution is optimal solution.

The optimal cost $= 20 \times 4 + 30 \times 11 + 50 \times 0 +$
$20 \times 0 + 10 \times 14 + 20 \times 6 + 30 \times$

$= ₹\ 1150/-$

Network scheduling by PERT / CPM :-

Basic planning and control planning that utilize a network to compute.

PERT is a probablistic method where the activities are represented by probability distribution.

PERT Procedure :-

1 : draw the project Network.

2. compute the expected duration of each activity using $t_e = \frac{t_0 + 4t_m + t_p}{6}$, Also calculat

te. expected variance (σ^2) of each activi

(ES)

3. compute earliest starting and earliest finishing (EF), Latest start (LS), Latest fin (LF), Total float (TF) of each activity.

4. Find the critical path and identify critic activities.

5. compute the project length and variance. $(\sigma^2$ and $\sigma)$

6. calculate Standard Normal Variable. (z)

$$z = \frac{T_s - T_e}{\sigma}$$

T_s - schedule time to complete the project.

Bhuvi

T_e - ~~Normal~~ expected project length duration.

σ - standard deviation (expected) of project length.

Note :

Earliest starting (ES) time denoted by □.

Latest finishing (LS) time " " △.

1. Following table shows the jobs of a network along with their time estimates

Job	1-2	1-6	2-3	2-4	3-5	4-5	6-7	5-6	7-8
a(days) t_o	1	2	2	2	7	5	5	3	8
m(") t_m	7	5	14	5	10	5	8	3	17
b(") t_p	13	14	26	8	19	17	29	9	32

Draw the project Network and find the probability that the project is completed in 40 days.

To obtain Exp. time & S.D for the activity

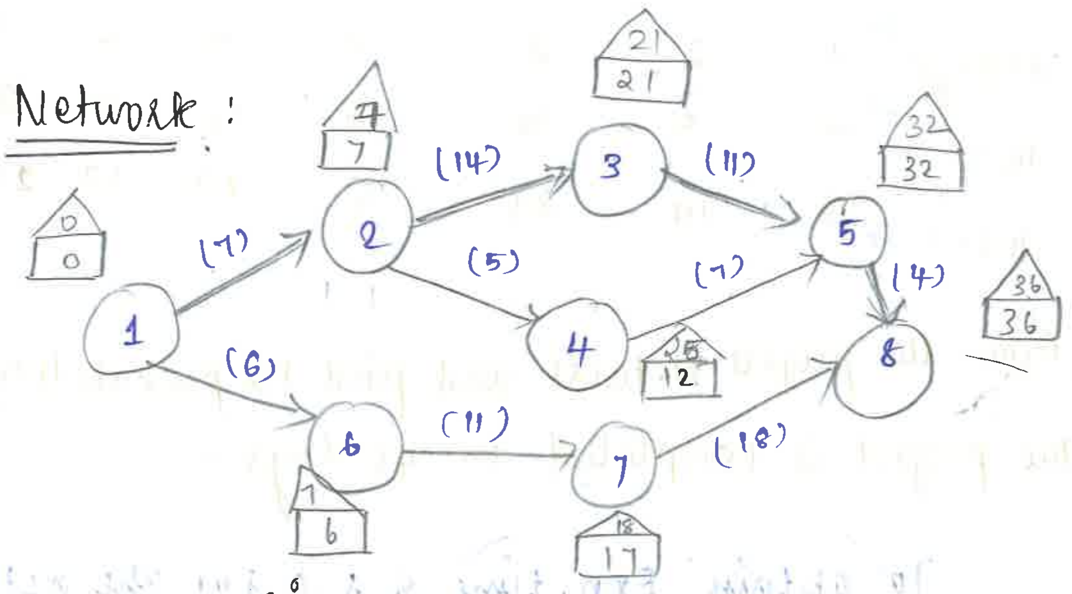
Activity ⇒	$t_e = \frac{t_o + 4t_m + t_p}{6}$,	$\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$
1-2	$\frac{1+4(7)+13}{6} = 7$	$\left(\frac{13-1}{6}\right)^2 = 4$
1-6	$\frac{2+4(5)+14}{6} = 6$	$\left(\frac{14-2}{6}\right)^2 = 4$

14
$\cancel{8}4/6$

7
9
76

2 - 3	$\frac{2+4(14)+26}{6} = 14$	$\left(\frac{26-2}{6}\right)^2 = 16$
2 - 4	$\frac{2+4(5)+8}{6} = 5$	$\left(\frac{8-2}{6}\right)^2 = 1$
3 - 5	$\frac{\cancel{7}+4(10)+19}{6} = 11$	$\left(\frac{19-7}{6}\right)^2 = 4$
4 - 5	$\frac{5+4(5)+17}{6} = 42$	$\left(\frac{17-5}{6}\right)^2 = 4$
6 - 7	$\frac{5+4(8)+29}{6} = 11$	$\left(\frac{29-5}{6}\right)^2 = 16$
5 - 8	$\frac{3+4(8)+9}{6} = 4$	$\left(\frac{9-3}{6}\right)^2 = 1$
7 - 8	$\frac{8+17(4)+32}{6} = 18$	$\left(\frac{32-8}{6}\right)^2 = 16$

Network :



Since distinct we will not move through it.

Expected duration project = 36 days.

critical path : 1 - 2 - 3 - 5 - 8.

project length variance (σ^2). = 4 + 16 + 1

= 25.

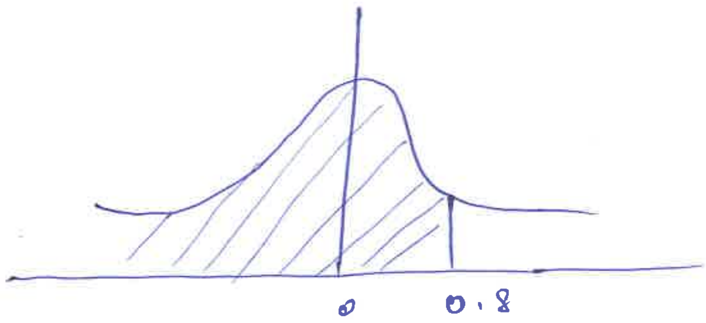
S.D, $\boxed{\sigma = 5}$.

Probability that the project will be completed in 40 days is given by

$$p(z \leq D), \quad D = \frac{T_S - T_e}{\sigma} = \frac{40 - 36}{5}$$

$$= \frac{4}{5} = 0.8$$

Area under the normal curve $p(z \nleq 0.8) =$



$= 0.5 + \phi(0.8)$

$= 0.5 + 0.2881$

$= 0.7881$

$= 78.81\%$

Conclusion : If the project is performed 100 times under same condition there will be 78.81 occasions for this job to be completed in 40 days.

S.D, $\boxed{\sigma = 5}$.

Probability that the project will be completed in 40 days is given by

$$p(z \leq D), \quad D = \frac{T_s - T_e}{\sigma} = \frac{40 - 36}{5}$$

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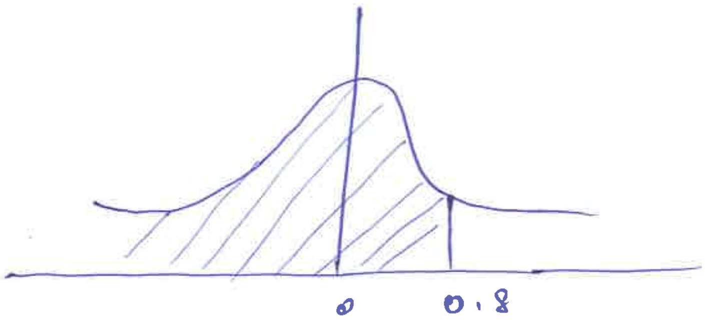
Area under the normal curve $p(z \leq 0.8) =$



$$= 0.5 + \phi(0.8)$$

$$= 0.5 + 0.2881$$

$$= 0.7881$$

$$= 78.81\%$$

Conclusion : If the project is performed 100 times under same condition there will be 78.81 occasions for this job to be completed in 40 days.

Construction of network :

1. A, B, C can start simultaneously.

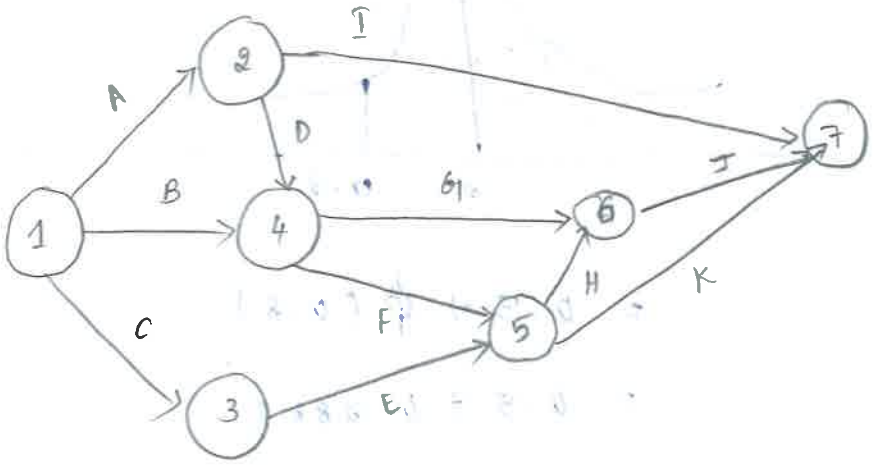
2m

A < D, I ; B < G, F ; C < E ; E < H, K ; F < H, K

G, H < J ; D < G, F .

soln :

Activity :	A	B	C	D	E	F	G	H	I	J	K
Predessor of activity :	-	-	-	A	C	B,D	B,D	E,F	A	G,H	E,F .



Floats and slack time :

CPM (critical path method)

Float is defined as the difference between the latest and earliest activity time.

Slack is defined as the difference between the latest and earliest event time .

Slack is only for events , Float is fo

activities.

There are three kinds of Floats.

* Total float (TF)

$$\boxed{TF = LS - ES}$$

LS - Latest start, earlier start -

2. Free float (FF) :

$$\boxed{FF = TF - HS}$$

HS - Head event slack

3. Independent Float (IF) :

$$\boxed{IF = FF - TS}$$

TS - Tail event slack.

NOTE :-

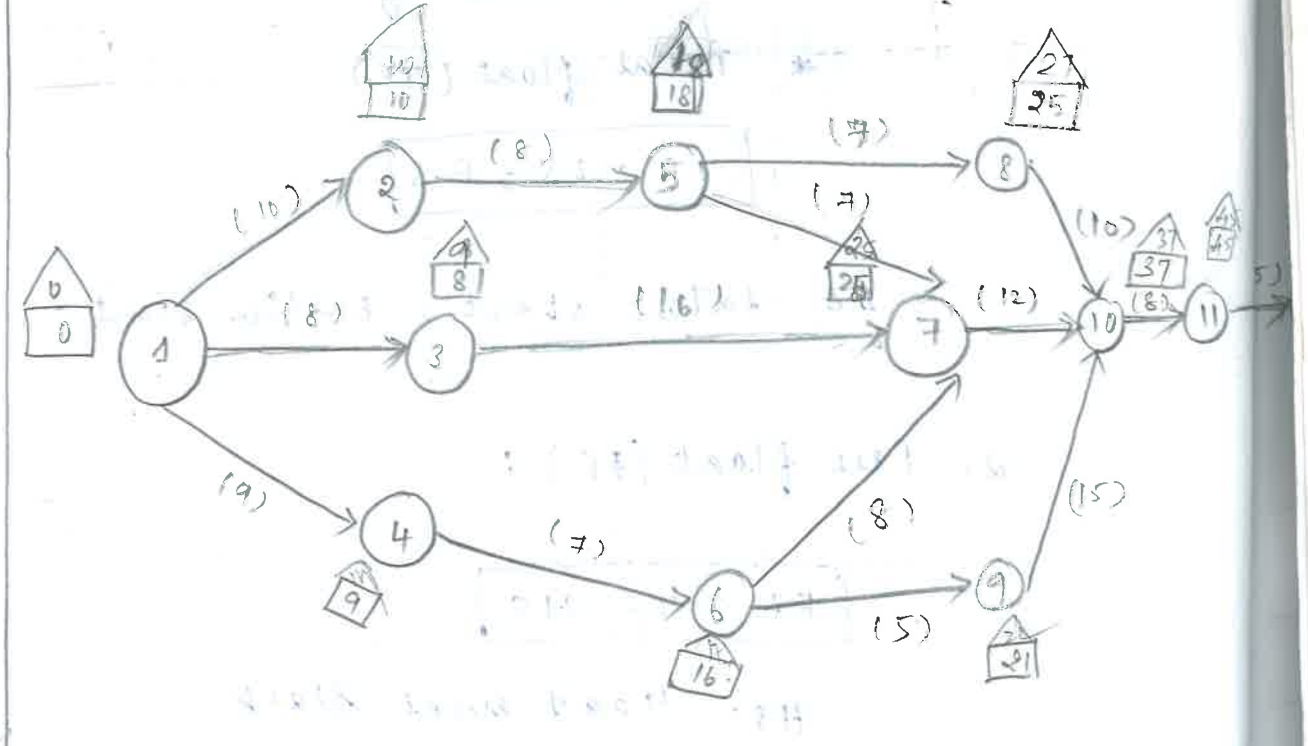
* A negative independent float is always taken as 0.

* Critical activities are obtained by where the TF is 0

The following table shows the job for their projects duration in days. Draw the network and determine the critical path and also calculate all the floats :

Jobs :	1-2	1-3	1-4	2-5	3-7	4-6	5-7	5-8	6-7	6-9	7-10	8-
Duration :	10	8	9	8	16	7	7	7	8	5	12	10

soln :



Critical path and floats :

Activity	time	Earliest		Latest		Floats		
		start	finish	start	finish	TF	FF	IF
1-2	10	0	10	0	10	0	0	0
1-3	8	0	8	1	9	1	0	0
1-4	9	0	9	1	10	1	0	0
2-5	8	10	18	10	18	0	0	0
3-7	16	8	24	9	25	1	1	0
4-6	7	9	16	10	17	1	0	0, when it is negative consid (−1)
5-7	7	18	25	18	25	0	0	0-0=0
5-8	7	18	25	20	27	2	0	0-0=0
6-7	8	16	24	17	25	1	1	1-1=0
6-9	15	16	21	17	22	[illegible]	0	0-1=0(=0
7-10	12	25	37	25	37	0	0	0-1=0

9-10	10-11	11-12
15	8	5

⑫

Activity	time	Earliest		Latest		Floats		
		Start	finish	Start	finish	TF	FF	IF
8-10	10	25	35	27	37	2	2	0
9-10	15	21	36	22	37	1	1	0
10-11	8	37	45	37	45	⓪	0	0
11-12	5	45	50	45	50	⓪	0	0

From the tables the TF of 1-2, 2-5, 5-7, 7-10, 10-11, 11-12 are 0. hence these are the critical activities.

critical path is 1-2-5-7-10-11-12

with the total project duration as 50 days.

---

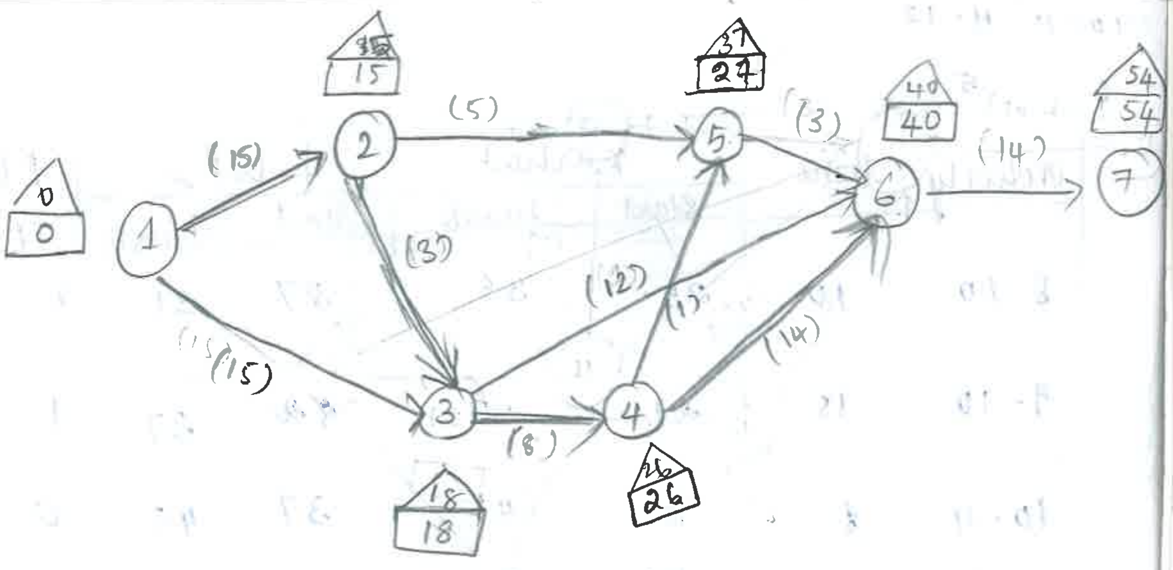
2. A small maintenance project consists of following jobs, whose precedence relationship is given below.

Jobs:	1-2	1-3	2-3	2-5	3-4	3-6	4-5	4-6	5-6	6-7
Duration: (Days)	15	15	3	5	8	12	1	14	3	14

a) Draw an Arrow Diagram representing project.

b) Find the total float for each Activity.

c) Find the critical path and total project duration.



Activity	Time	Earliest		Latest		Floats		
		Start	Finish	start	finish	TF	FF	I
1 - 2	15							
1 - 3	15							
2 - 3	3							
2 - 5	5							
3 - 4	8							
3 - 6	12							
4 - 5	1							
4 - 6	14							
5 - 6	3							
6 - 7	14							

22/09/22

DECISION MODELS

Decision making is an everyday process in life (day - to day). Decisions may classified into two categories.

i) Tactical and

ii) Strategic

Tactical Decisions :

Are those which affects the bussiness in short run.

Strategic Decisions :

Are those which have far reaching effects on the course of bussiness.

Types of decision making situations :

1. Decision making under certainly :

Decisions makers knowns the consequences of every choice.

2. Decision making under uncertainly :-

New product introduced in the market.

3. Decision making under risk :

By experience optimise the expected profit and minimum loss.

4. Decision making under conflict :

Two or more persons desire a particular thing and they compete each other.

Game theory :

Game theory deals with mathematica analysis of competetive problems it is a decision theory applicable to competetive situation where there are two or more opposing parties with conflicting interest. Such a competetive situation is called a Game.

It has the following properties.

- * There are finite no. of competitors called players.
- * player A - maximising player

  Player B - minimizing player.

Two person zero-sum games :

The algebric sum of gains and losses of all players is zero in game is called zero-sum game otherwise non-zero sum game.

Pure strategies :

It is a decision rule always to select a particular course of action.

Pure strategies are called optimum

strategies. (Max or Min)

If $\boxed{\text{Maxmin value} = \text{Minmax value}}$, then the payoff of matrix is saddle point that is the value of the game denoted by 'V'.

NOTE :

1) If both of them of the not equal Maxmin $\neq$ Minmax, then it is mixed strategy.

2) saddle point need not be unique.

Q/

1. Solve the following game whose payoff matrix is

$$\begin{array}{c|c|c}
 & \text{B's strategy} & \text{min} \\
\text{A's strategy} & \begin{pmatrix} 9 & 3 & 1 & 8 & 0 \\ 6 & 5 & 4 & 6 & 7 \\ 2 & 4 & 3 & 3 & 8 \\ 5 & 6 & 2 & 2 & 1 \end{pmatrix} & \begin{array}{c} 0 \\ \boxed{4} \rightarrow \text{max} \\ 2 \\ 1 \end{array} \\
\text{max} & \begin{array}{ccccc} 9 & 6 & \underset{\text{min}}{\boxed{4}} & 8 & 8 \end{array} &
\end{array}$$

Soln :

$$\begin{array}{c|c|c}
 & & \text{min} \\
\text{A's} & \begin{pmatrix} 9 & 3 & 1 & 8 & 0 \\ 6 & 5 & 4 & 6 & 7 \\ 2 & 4 & 3 & 3 & 8 \\ 5 & 6 & 2 & 2 & 1 \end{pmatrix} & \begin{array}{c} 0 \\ \boxed{4} \rightarrow \text{max} \\ 2 \\ 1 \end{array} \\
\text{max} & \begin{array}{ccccc} 9 & 6 & \underset{\text{min}}{\boxed{4}} & 8 & 8 \end{array} &
\end{array}$$

Max. of Row minima = 4 = Min. of col maxima

A's pure strategy A_2

B's pure strategy B_3

∴ value of the game $\boxed{V = 4}$.

Mixed strategies [games without saddle point].

For 2 x 2 two person zero sum pay off matrix is

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & a & b \\ A_2 & c & d \end{array}$$

i) The optimum mixed strategies

$$S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix} \text{ and } S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$$

where $p_1 = \frac{d-c}{\lambda}$; $p_2 = 1 - p_1$ [$\therefore p_1 + p_2 =$

$q_1 = \frac{d-b}{\lambda}$; $q_2 = 1 - q_1$

and, $\lambda = (a+d) - (b+c)$

game value $\boxed{v = \frac{ad-bc}{\lambda}}$.

2. Solve $\begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} \begin{array}{c} \text{Mi} \\ -1 \\ -1 \end{array}$

soln:

$$\begin{array}{c|cc|c} & & & \text{min} \\ & 2 & -1 & \textcircled{-1} \to \text{max} \\ & -1 & 0 & -1 \\ \text{Max} & 2 & \textcircled{0}\ \text{min} & \end{array}$$

This is a game with out saddle point

∴ [Minmax ≠ Max min ≠ 0 ≠ -1]

∴ By mixed strategy

$$\lambda = (2+0) - (-1+(-1)) = 2+2 = 4$$

$$v = \frac{(2)(0) - (-1)(-1)}{4} = -1/4$$

$P_1 = \frac{0+1}{4} = 1/4$; $P_2 = 1 - 1/4 = 3/4$ [∴ $P_1 + P_2 = 1$]

$q_1 = \frac{0+1}{4} = 1/4$; $q_2 = 1 - 1/4 = 3/4$

∴ A's strategy / $S_A = \begin{pmatrix} A_1 & A_2 \\ 1/4 & 3/4 \end{pmatrix}$

B's strategy / $S_B = \begin{pmatrix} B_1 & B_2 \\ 1/4 & 3/4 \end{pmatrix}$.

7) In a game of matching coins with 2 players supposed A wins with 1 unit value when there are two heads, wins nothing when there are two tails and looses 1/2 unit value, when there are one Head and tail and, determines the play off matrix for the best strategy each player and the value of the game.

soln:

$$\begin{array}{c|cc|c} & \overset{a}{H} & \overset{b}{T} & \text{Min} \\ \hline H & 1 & -1/2 & \textcircled{-1/2} \\ & {\scriptstyle c} & {\scriptstyle d} & \\ T & -1/2 & 0 & -1/2 \\ \hline \text{Max} & 1 & \textcircled{0} & \end{array}$$

MinMax ≠ Max Min

ie): mixed strategy.

$\lambda = ((1)+(0)) - (-1/2 - 1/2)$

$= 1+1$

$= 2$

$P_1 = \dfrac{0+\frac{1}{2}}{2} = \dfrac{1}{4} \Rightarrow P_2 = 1 - P_1$

$= 3/4$.

$q_1 = \dfrac{0+1/2}{2} = 1/4 \Rightarrow q_2 = 1 - q_1$

$= 3/4$

value of the game, $v = \dfrac{ad - bc}{\lambda}$

$= \dfrac{0 - 1/4}{2} = -1/$

A's strategy $S_A = \begin{pmatrix} A_1 & A_2 \\ 1/4 & 3/4 \end{pmatrix}$

B's strategy $S_B = \begin{pmatrix} B_1 & B_2 \\ 1/4 & 3/4 \end{pmatrix}$

Dominance property :

To bring 2 x 2 square matrix from rectangle matrix with order more than 2, remove dominant column and lesser rows.

1. Solve the following game using dominance property

$$\begin{array}{c|ccc} & I & II & III \\ \hline I & 1 & 7 & 2 \\ II & 6 & 2 & 7 \\ III & 6 & 1 & 6 \end{array}$$

Row - less
Column - greater } remove

Soln :

$\therefore R_{III} \leq R_{II}$ omit R_{III}

$$\begin{array}{c|ccc} & I & II & III \\ \hline I & 1 & 7 & 2 \\ II & 6 & 2 & 7 \end{array}$$

$C_I \leq C_{III}$

omit C_{III}

$$\begin{array}{c|cc|c} & I & II & \text{Min} \\ \hline I & 1 & 7 & 1 \\ II & 6 & 2 & \boxed{2} \\ \hline \text{Max} & \boxed{6} & 7 & \end{array}$$

$\lambda = (1+2) - (7+6)$

$= 3 - 13 \quad = -10$

$P_1 = \frac{2-6}{10} = \frac{4}{10} = 2/5 \therefore P_2 = 3/5$

$q_1 = \frac{2-7}{10} = 5/10 = 1/2 \quad \therefore q_2 = 3/2$

value of the game, $v = \frac{2 - 42}{-10}$

$= \cancel{-}\frac{4\cancel{0}}{\cancel{-}1\cancel{0}}$

$\boxed{V = 4}$

Mixed strategies

$S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 2/5 & 3/5 & 0 \end{pmatrix} \qquad S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 1/2 & 1/2 & 0 \end{pmatrix}$

Elimnate least m max m col

2) Using Dominance property solve.

$$A's \; \begin{array}{c|cccc} & I & II & B's\; III & IV \\ \hline I & -5 & 3 & 1 & 20 \\ II & 5 & 5 & 4 & 6 \\ III & -4 & -2 & 0 & -5 \end{array}$$

soln :

$R_{\underline{III}} \leq R_{\underline{II}}$

omit $R_{\underline{III}}$

$$\begin{array}{c} I \\ 2 \end{array}\begin{pmatrix} -5 & 3 & 1 & 20 \\ 5 & 5 & 4 & 6 \end{pmatrix}$$

$C_{III} \leq C_{IV}$

omit C_{IV}

$$\begin{pmatrix} -5 & 3 & 1 \\ 5 & 5 & 4 \end{pmatrix}$$

$C_{I} \leq C_{II}$

omit C_{II}

$$\begin{array}{cccc} & I & III & \text{Min} \\ I & -5 & 1 & -5 \\ II & 5 & 4 & \boxed{4} \\ \text{Max} & 5 & \boxed{4} & \end{array}$$

Minmax = Maxmin (saddle point)

Max of row minima = 4 = min of col maxima

A's pure strategy . A_2

B's pure strategy , B_3 .

Matrix $\underbrace{\text{oddment}}_{\text{simplifying}}$ method for nxn games : [Arithmetic method]

step 1 : obtain a new matrix 'c' from $A = [a_{ij}]$

by $c_1 \Leftrightarrow c_1 - c_2$ and $c_2 \leftrightarrow c_2 - c_3$

step 2 : obtain a new 'R' from 'A' by

$R_1 \Leftrightarrow R_1 - R_2$

and $R_2 \leftrightarrow R_2 - R_3$

step 3 : Determine the magnitude of oddment that is obtained $|c_i|$ by deleting ith row. and $|R_j|$ by deleting jth colu

step 4 : write the magnitude of oddments. (without (–) sign against their row and columns).

step 5 : The oddments expressed as fractions of the grand total yield optimum stratgy when sum of rows an columns oddments are equal. (If not method fails).

step 6 : Calculate the expected value of t game with the optimum mixed strategy for the row player [c_1 -wise]

1. Solve the following 3 x 3 game by the method of oddmen's

$$\begin{array}{cc} & \text{player B} \\ \text{Player A} & \begin{bmatrix} 3 & -1 & -3 \\ -3 & 3 & -1 \\ -4 & -3 & 3 \end{bmatrix} \end{array}$$

soln : By oddments method.

$$C = \left[\begin{array}{cc} 4 & 2 \\ -6 & 4 \\ -1 & -6 \end{array}\right]$$

$$\begin{array}{cc} C_1 - C_2 & C_2 - C_3 \end{array}$$

$$R = \left[\begin{array}{ccc} 6 & -4 & -2 \\ 1 & 6 & -4 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ R_2 - R_3 \end{array}$$

$$\text{Now } |C_1| = \begin{vmatrix} -6 & 4 \\ -1 & -6 \end{vmatrix} \quad \text{omit } R_1$$

$$= 36 + 4 = 40$$

$$|C_2| = \begin{vmatrix} 4 & 2 \\ -1 & -6 \end{vmatrix} \quad \text{omit } R_2.$$

$$= -24 + 2 \Rightarrow -22 \sim$$

$$|C_3| = \begin{bmatrix} 4 & 2 \\ -6 & 4 \end{bmatrix} \quad \text{omit } R_3$$

$$= 16 + 12 = 28$$

omit C_1 $$|R_1| = \begin{bmatrix} -4 & -2 \\ 6 & -4 \end{bmatrix}$$

$$= +16 + 12 = 28$$

omit C_2 $$|R_2| = \begin{vmatrix} 6 & -2 \\ 1 & -4 \end{vmatrix} = -24 + 2 = -22$$

omit C_3 $$|R_3| = \left|\begin{array}{cc} 6 & -4 \\ 1 & 6 \end{array}\right] = 36 + 4 = 40.$$

The augumented payoff matrix is

$$\begin{array}{lcccc} & & & & \text{Row oddments} \\ & 3 & -1 & -3 & 40 \\ & -3 & 3 & -1 & 22 \\ & -4 & -3 & 3 & 28 \\ \text{column oddments} & 28 & 22 & 40 & \boxed{90} \end{array}$$

Optimum Strategies are :

i) A's strategy, $S_A = \left(\frac{40}{90}, \frac{22}{90}, \frac{28}{90}\right)$

$= \left(\frac{4}{9}, \frac{11}{45}, \frac{14}{45}\right)$

ii) B's strategy $S_B = \left(\frac{28}{90}, \frac{22}{90}, \frac{40}{90}\right)$

$= \left(\frac{14}{45}, \frac{11}{45}, \frac{4}{9}\right)$

Value of the game .

[For player A by c_1-wise] $= \frac{4}{\cancel{9}_{3}} \times \cancel{3} + \frac{11}{\cancel{45}_{15}} \times -\cancel{3} + \frac{4}{4}$

$= \frac{4}{3} + \frac{(-11)}{15} + \frac{-56}{45}$

$= \frac{4}{3} - \frac{33-56}{45}$

$= \frac{4}{3} - \left(\frac{33+56}{45}\right)$

$= \frac{4}{3} - \frac{89}{45} = \frac{60-}{4}$

$= -29/45.$

2. Solving the 3×3 game by method of matrices.
oddment :

$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

Form new matrices c and R.

soln : By oddments method.

$$C = \begin{bmatrix} 2 & 0 \\ 0 & -4 \\ -3 & 3 \end{bmatrix} \quad R = \begin{bmatrix} 2 & 0 & -4 \\ 0 & -3 & 4 \end{bmatrix}$$

$$|C_1| = \begin{vmatrix} 0 & -4 \\ -3 & 3 \end{vmatrix} = 0-(+12) = -12$$

$$|C_2| = \begin{vmatrix} 2 & 0 \\ -3 & 3 \end{vmatrix} = 6-0 = 6$$

$$|C_3| = \begin{vmatrix} 2 & 0 \\ 0 & -4 \end{vmatrix} = -8+0 = -8$$

$$|R_1| = \begin{vmatrix} 0 & -4 \\ -3 & 4 \end{vmatrix} = 0-(+12) = -12$$

$$|R_2| = \begin{vmatrix} 2 & -4 \\ 0 & 4 \end{vmatrix} = 8-0 = 8$$

$$|R_3| = \begin{vmatrix} 2 & 0 \\ 0 & -3 \end{vmatrix} = -6+0 = -6$$

The augmented pay off matrix is.

$$\begin{array}{lcccc} & & & & \text{Row oddments} \\ & 1 & -1 & -1 & 12 \\ & -1 & -1 & 3 & 6 \\ & -1 & 2 & -1 & 8 \\ \text{column oddments} & 12 & 8 & 6 & \boxed{26} \end{array}$$

Optimum strategies are

i) A's strategy $S_A = \left(\frac{12}{26}, 6/26, 8/26\right)$

$= \left(6/13, 3/13, 4/13\right)$

ii) B's strategy $S_B = \left(12/26, 8/26, 6/26\right)$

$= \left(6/13, 4/13, 3/13\right)$

Value of game

[For player A with $\underline{\underline{c_1\text{-wise}}}$]

$= 6/13 \times 1 + \frac{3}{13} \times -1 + \frac{4}{13} \times -$

$= \frac{6}{13} - \frac{3}{13} - \frac{4}{13}$

$= \frac{-1}{13}$.

Type 3 : Graphical method [2×n or m×2 games]

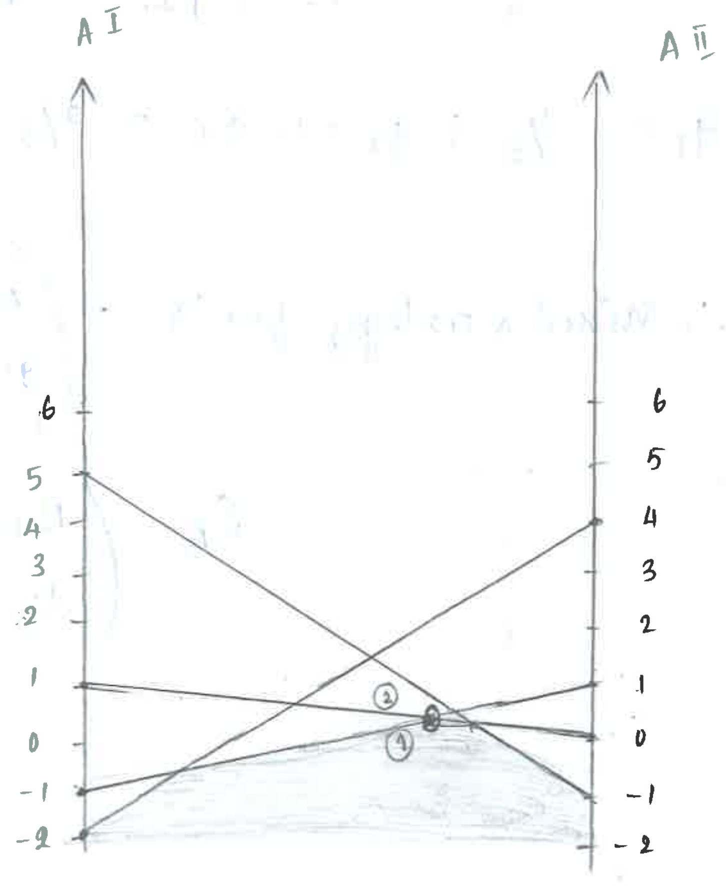
For 2×2 payoff matrix, player A has to reach the highest point on the lower boundary.

For m×2 games get minmax point in between the lowest point on the upper boundary.

1. Solve the following 2×4 game graphically

$$\text{player } Ⓐ \begin{array}{c} \\ \end{array} \underset{}{\overset{\text{player } Ⓑ}{\begin{pmatrix} \overset{I}{1} & \overset{II}{0} & 4 & -1 \\ -1 & 1 & -2 & 5 \end{pmatrix}}} \begin{array}{c} A\,II \\ A\,I \end{array}$$

soln: Consider 2 Accers, say AI & AII vertically at unit distance apart.



Max min lines ① & ②

$$\begin{array}{c|cc|c} & & & \text{Min} \\ & 1 & 0 & 0 \\ & -1 & -1 & -1 \\ \hline \text{Max} & 1 & 1 & \end{array}$$

Min max ≠ maxmin [Not saddle point]

ie, mixed strategy

$$\lambda = \overset{a+d}{(1+1)} - \overset{b+c}{(0-1)} = 3$$

$$\text{Value of the game} \Rightarrow V = \frac{(ad)-(bc)}{\lambda}$$

$$= \frac{1}{3}.$$

$$p_1 = \frac{(1+1)}{3} = 2/3 \;;\; p_2 = 1 - p_1 = 1/3.$$

$$q_1 = 1/3 \;;\; q_2 = 1 - q_1 = 2/3.$$

$$\therefore \text{Mixed strategy for } S_A = \begin{pmatrix} A_1 & A_2 \\ 2/3 & 1/3 \end{pmatrix}$$

$$S_B = \left(\begin{matrix} B_1 & B_2 & B_3 & B_4 \\ 1/3 & 2/3 & 0 & 0 \end{matrix} \right.$$

2) Use the notion of dominance who simplify the rectangular game with the following payoff and solving graphically.

Player K

$$\begin{array}{cc|cccc} & & I & \overline{II} & \overline{III} & \overline{IV} \\ \hline \text{player L} & 1 & 18 & 4 & 6 & 4 \\ & 2 & 6 & 9 & 13 & 7 \\ & 3 & 11 & 5 & 17 & 3 \\ & 4 & 7 & 6 & 12 & 2 \end{array} \quad 4\times4$$

2×n
m×2

2×2

max c]
min

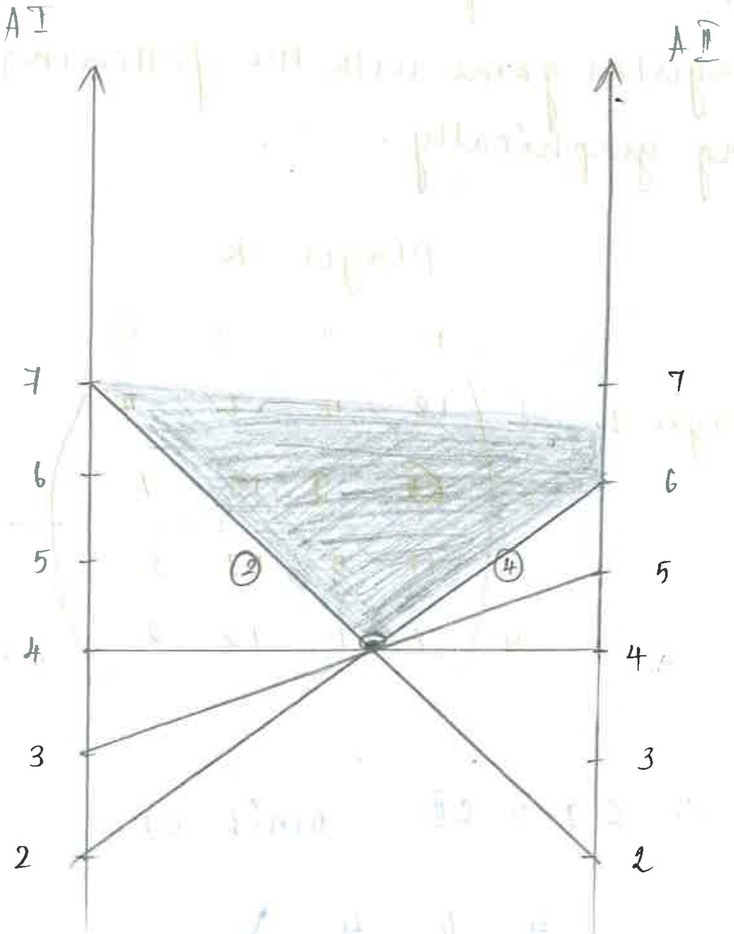
Soln: ∴ C I > C $\overline{II}$ omit C I

$$\begin{pmatrix} 4 & 6 & 4 \\ 2 & 13 & 7 \\ 5 & 17 & 3 \\ 6 & 12 & 2 \end{pmatrix}$$

C $\overline{III}$ > C$\overline{II}$ omit C$\overline{III}$

$$\begin{array}{c|cc} & ② & ④ \\ \hline 1 & 4 & 4 \\ 2 & 2 & 7 \\ 3 & 5 & 3 \\ 4 & 6 & 2 \\ \hline & A\overline{II} & AI \end{array}$$

mx2



$$\begin{array}{cccc} & \text{ii} & \text{iv} & \text{Min} \\ 2 & 2 & 7 & 2 \\ 4 & 6 & 2 & 2 \\ \text{Max} & 6 & 7 & \end{array}$$

Maxmin ≠ Minmax

ie mixed strategy

$\lambda = (2+2) - (6+7) = -9$

Game value $V = \frac{(2)(2) - (6)(7)}{-9} = 38/9$

$P_1 = \frac{2-6}{-9} = 4/9 \Rightarrow p_2 = 5/9$

$q_1 = \frac{2-7}{-9} = 5/9 \Rightarrow q_2 = 4/9$

Mixed strategies :

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 4/9 & 0 & 5/9 \end{pmatrix}$$

$$S_B = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 5/9 & 0 & 4/9 \end{pmatrix}$$

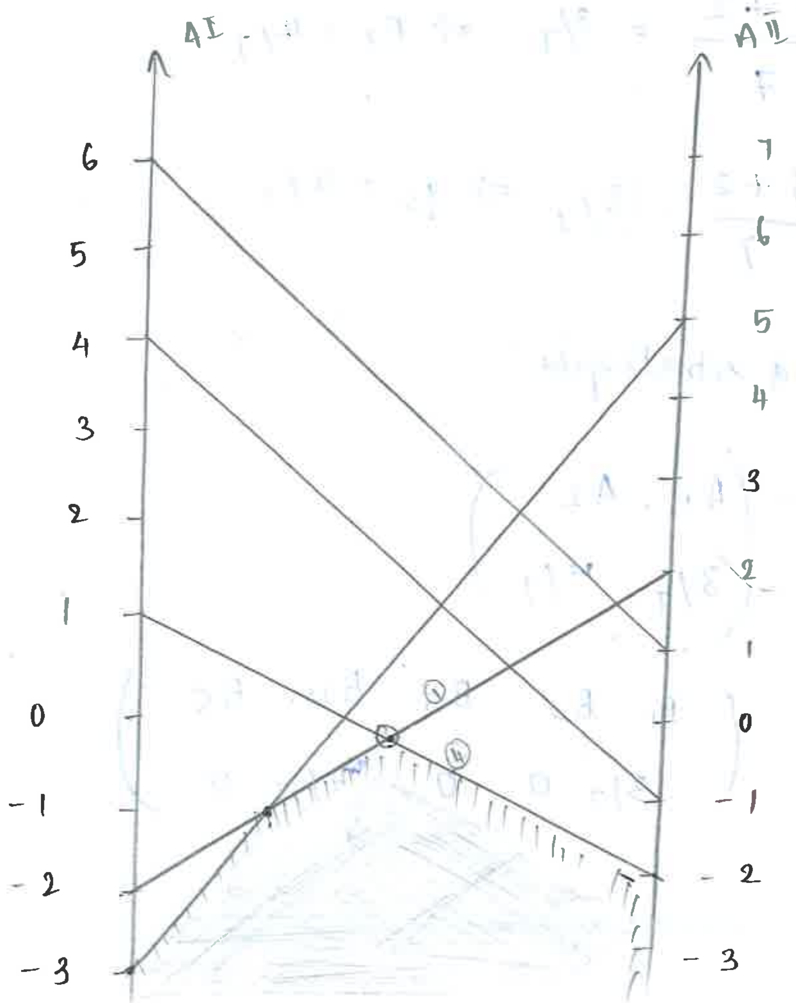
---

3. Using graphical method, solve the rectangular game, for player A is

2 × n, lower
m × 2 upper

$$\begin{array}{cl} \begin{pmatrix} 2 & -1 & 5 & -2 & 6 \\ -2 & 4 & -3 & 1 & 0 \end{pmatrix} & \begin{array}{l} A_{II} \\ A_I \end{array} \end{array}$$

Soln : considering A_{II}, A_I.



payoff matrix is

$$\begin{array}{cccc} & \underline{I} & \underline{IV} & \text{Min} \\ 1 & \overset{a}{2} & \overset{b}{-2} & -2 \\ 2 & \overset{c}{-2} & \overset{d}{1} & \boxed{-2} \\ \text{Max} & 2 & \boxed{1} & \end{array}$$

Minmax ≠ Maxmin

∴ Mixed strategies

$\lambda = (2+1) - (-2-2) = 7$

Value of the game $v = \dfrac{(2)(1) - (-2)(-2)}{7} =$

$P_1 = \dfrac{1+2}{7} = 3/7 \Rightarrow P_2 = 4/7$

$q_1 = \dfrac{1+2}{7} = 3/7 \Rightarrow q_2 = 4/7$.

Mixed strategies:

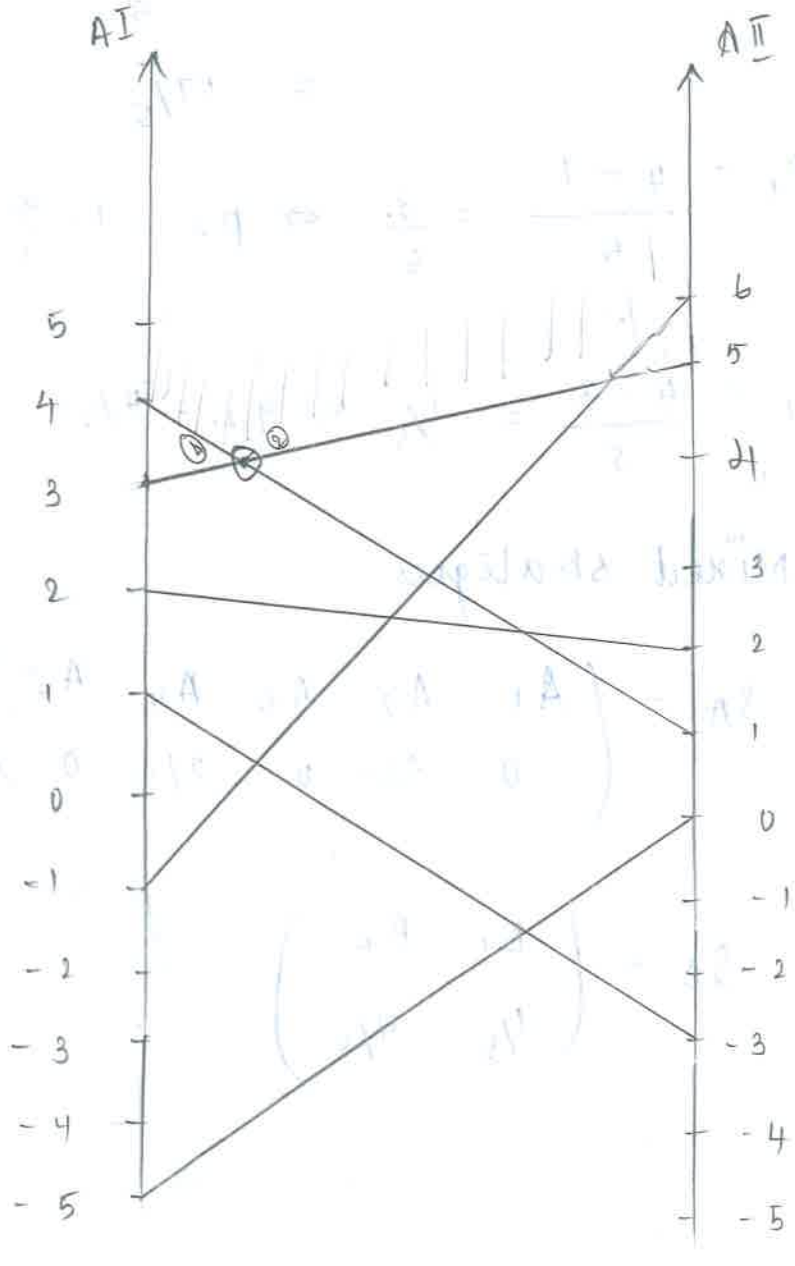
$$S_A = \begin{pmatrix} A_1 & A_2 \\ 3/7 & 4/7 \end{pmatrix}$$

$$S_B = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 & B_5 \\ 3/7 & 0 & 0 & 4/7 & 0 \end{pmatrix}$$

4. Solve the following game using graphical method

$$\begin{array}{cc} & \text{player B} \\ \text{player A} & \begin{pmatrix} -3 & 1 \\ 5 & 3 \\ 6 & -1 \\ 1 & 4 \\ 2 & 2 \\ 0 & -5 \end{pmatrix} \\ & \begin{array}{cc} A\underline{II} & AI \end{array} \end{array}$$

soln: Considering AII, AI.



Payoff matrix is

$$\begin{array}{cc|c} & & \text{Min} \\ 2 & \begin{pmatrix} 5 & 3 \\ 1 & 4 \end{pmatrix} & \begin{array}{c} \boxed{3} \\ 1 \end{array} \\ 4 & & \\ \text{Max} & 5 \quad \boxed{4} & \end{array}$$

Minmax ≠ Maxmin

∴ Mixed strategies

$\lambda = (5+4) - (1+3) = 5$

Value of game, $V = \dfrac{(5)(4) - (1)(3)}{5} = \dfrac{20-3}{5}$

$= 17/5$.

$p_1 = \dfrac{4-1}{5} = \dfrac{3}{5} \Rightarrow p_2 = 1 - \dfrac{3}{5} = 2/5$

$q_1 = \dfrac{4-3}{5} = 1/5 \Rightarrow q_2 = 4/5$.

Mixed strategies

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ 0 & 3/5 & 0 & 2/5 & 0 \end{pmatrix}$$

$$S_B = \begin{pmatrix} B_1 & B_2 \\ 1/5 & 4/5 \end{pmatrix}$$

Replacement Model :-

Usually a system contains a large no. of. low cost items that are increasingly liable to fail with age. Very often the failure of an item may cause a total break down of the system, where we requires replacement of an item.

Eg : A tube or condenser in a aircraft costs little but its failure may total collapse of aircraft. A small component can bring huge failure.

Types of Replacements :

1. Individual Replacement :

An item is replaced immediatly after it fails.

2. Group Replacements :

All items are replaced after particular time interval.

Ex : Bridges.

1. Find the cost per period of individual replacement policy of a installation of 200 light bulbs, given the following

i) cost of replacing an individual bulb is ₹ 3.

ii) Conditional probability of failure is given below

Week No	0	1	2	3	4
conditional probability	0	0.1	0.2	0.7	1.0

Soln: The prob. that a bulb fails during the week.

$P_0 = 0$

$P_1 = 0.1$

$P_2 = 0.3 - 0.1 = 0.2$

$P_3 = 0.7 - 0.3 = 0.4$

$P_4 = 1 - 0.7 = 0.3$

$\therefore \Sigma p_i = 1$, all the probabilities of higher than P_4 is 0.

Let N_i be the no of replacements made.

$N_0 = 200$ [$\therefore$ Initially all 200 bulbs]

$N_1 = N_0 P_1$

$= 200 \times 0.1 = 20$

$N_2 = N_0 P_2 + N_1 P_1$

$= 200 \times 0.2 + 20 \times 0.1$

$= 42$

$N_3 = N_0 P_3 + N_1 P_2 + N_2 P_1$

$= 200 \times 0.4 + 20 \times 0.2 + 42 \times 0.1$

$= 88 \ (\therefore 88.2)$

$N_4 = N_0 P_4 + N_1 P_3 + N_2 P_2 + N_3 P_1$

$= 200 \times 0.3 + 20 \times 0.4 + 42 \times 0.2 + 88 \times 0.1$

$= 85 \quad (\because 84.8)$

Thus, the no of bulbs failing each week increases till 3rd week and decreases at 4th week.

$$\text{Avg life of bulbs} = \sum_{i=1}^{4} i p_i = 1 \times 0.1 + 2 \times 0.2 + 3 \times 0.4 + 4 \times 0.3$$

$$= 2.9 \text{ weeks}$$

$$\therefore \left.\begin{array}{l}\text{Avg no of failure}\\ \text{per week}\end{array}\right\} = \frac{200}{2.9} = 69 \quad (\because 68.9)$$

$\therefore$ cost of individual replacement of bulbs $=$ Rs. 3×69

= Rs. 207 per week.

---

2. Following mortality rate has been observed for a certain type of transition.

End of week :	1	2	3	4	5	6
Prob of failure :	0.09	0.25	0.49	0.85	0.97	1.00

There are large no. of. such transistors which all to be in working order. If the transistor fails in service, it costs Rs. 4 to replace. But if all the transistors replace in the same operation, It can be done only Rs. 0.80 a transistor. It is proposed to replace all transistors at fixed intervals whether not they have worked out. continue ~~eq~~ replacing work out as they fail.

i) What is the best interval between group replacements?

ii) At what group replacement peice the transistor put a policy of strictly individual replacement become refenable to the adopted policy.

Soln: Let us assume that no. of. transistors whose B = 1000 and P_i be the probability of a new transistor faits i^{th} week.

$P_1 = 0.09$

$P_2 = 0.25 - 0.09 = 0.16$

$P_3 = 0.49 - 0.25 = 0.24$

$P_4 = 0.85 - 0.49 = 0.36$

$P_5 = 0.97 - 0.85 = 0.12$

$P_6 = 1 - 0.97 = 0.03$

ie, All the probabilities more than P_6 must vanish is 0.

∴ All transistors are worked out by the 6th week.

Let N_i be the no. of. Replacements made, when 1000 transistors are new initially.

ie, $N_0 = 1000$

$N_1 = N_0 P_1 = 1000 \times 0.09 = 90$.

$N_2 = N_0 P_2 + N_1 P_1 = 1000 \times 0.16 + 90 \times 0.09$

$= 168$.

$N_3 = N_0 P_3 + N_1 P_2 + N_2 P_1 = 1000 \times 0.24 + 90 \times 0.16 + 168 \times 0.09$

$= 269$.

$N_4 = N_0 P_4 + N_1 P_3 + N_2 P_2 + N_3 P_1$

$= 1000 \times 0.36 + 90 \times 0.24 + 168 \times 0.16 + 269 \times 0.09$.

$= 432$

$N_5 = N_0 P_5 + N_1 P_4 + N_2 P_3 + N_3 P_2 + N_4 P_1$

$= 1000 \times 0.12 + 90 \times 0.36 + 168 \times 0.24 + 269 \times 0.16 + 432 \times 0.09$

$= 274$.

$N_6 = N_0 P_6 + N_1 P_5 + N_2 P_4 + N_3 P_3 + N_4 P_2 + N_5 P_1$.

$= 1000 \times 0.03 + 90 \times 0.12 + 168 \times 0.36 + 269 \times 0.24 + 432 \times 0.16 + 274 \times 0.09$.

$= 259$.

$N_7 = N_0 P_7 + N_1 P_6 + N_2 P_5 + N_3 P_4 + N_4 P_3 + N_5 P_2 + N_6 P_1$

$= 1000 \times (0) + 90 \times 0.03 + 168 \times 0.12 + 269 \times 0.36 + 432 \times 0.24 + 274 \times 0.16 + 259 \times 0.09$.

$= 290$.

∴ No. of Transistors failing each week increases till fourth week, then decreases and again increases at 7th week.

∴ Replacement requires oscillately.

i) Individual replacement :

Avg. life of transistors $= \sum_{i=1}^{6} i P_i$

$= 1P_1 + 2P_2 + 3P_3 + 4P_4 + 5P_5$
$6P_6$

$= 0.09 + 2 \times 0.16 + 3 \times 0.24 +$
$4 \times 0.36 + 5 \times 0.12 + 6 \times 0.$

$= 3.35$,

Avg. no. failure per week $= \frac{1000}{3.35} = 299$ ($\because$ 298.50)

∴ cost per week of individual replacement of transistors } Rs = 4 × 299
Rs : 1196/-

Under group replacement :

∴ The replacement of all 1000 balls in operation costs 4 Rs. 0.80 per bulb and individu bulb costs Rs. 4 the total costs of replacemen is

End of week	Tot cost under group replacement (Rs)	Avg cost per week
1.	1000×0.80 = 800	800/1 = 8
2.	(1000×0.80) + (90×4) = 1160	1160/2 = 5
3.	(1000×0.80) + (90×4) + (168×4) = 1832	1832/3 = 6
4.	(1000×0.80) + (90×4) + (168×4) + (269×4) = 2908	$\frac{2908}{4}$ = 72

a) Average min^m cost occur   It is 101 optimal to have group replacement every two weeks but in individual every 4th week we need replacement. Hence according to the individual replacement is preferred.

b) Let x be the price of a bulb in group replacement then Rs. 1196 < $\frac{1000x + (4 \times 90)}{2}$.

$$1196 < \frac{1000x + (4 \times 90)}{2}$$

$$1196 < 500x + 180$$

$$1016 < 500x$$

$$x \geq \frac{1016}{500}$$

$$x > Rs.\ 2.03\ .$$

3. The following mortality rates have been observed for a certain type of condensers

Week	1	2	3	4	5
% of failing by the end of week	10	25	50	80	100

There are 1000 condenser in use and its cost ₹1, Replaced an individual condenser which has burnt out. If all condenses were replaced simultaneously. It would cost ₹ 25 paise per condenser. It is proposed to replace all condenser at finial interval whether are not they have burnt out and to continue replacing burnt out condenser as their faults. At what intervals should all the condenser replaced.

Soln: Let p_i be the probability of failure in i^{th} week.

$$P_1 = \frac{10}{100} = 0.1$$

$$P_2 = \frac{25-10}{100} = \frac{15}{100} = 0.15$$

$$P_3 = \frac{50-25}{100} = 0.25$$

$$P_4 = \frac{80-50}{100} = 0.30$$

$$P_5 = \frac{100-80}{100} = 0.20$$

$$P_6 = P_7 = = 0$$

∴ The condenser cannot survive for more than 5 weeks.

Let N_i i^{th} week.

Let $N_0 = 1000$

$$N_1 = N_0 P_1 = 1000(0.1) = 100$$

$$N_2 = N_0 P_2 + N_1 P_1 = 1000 \times 0.15 + 100 \times 0.1$$
$$= 150 + 10 = 160$$

$$N_3 = N_0 P_3 + N_1 P_2 + N_2 P_1$$
$$= 1000 \times 0.25 + 100 \times 0.15 + 160 \times 0.1$$
$$= 250 + 15 + 16 = 281$$

Model IV : (M/M/s) : (N/FCFS)

[This model is as Model II, except N > s]

1. $\lambda_n = \begin{cases} \lambda, & 0 \leq n \leq N \\ 0, & n > N \end{cases}$

2. $\mu_n = \begin{cases} n\mu, & 0 \leq n \leq s \\ s\mu, & s < n \end{cases}$

3. $P_n = \begin{cases} \frac{1}{n}\left(\frac{\lambda}{\mu}\right)^n P_0, & 0 \leq n < s \\ \frac{1}{s^{n-s}s!}\left(\frac{\lambda}{\mu}\right)^n P_0, & s \leq n \leq N \end{cases}$

4. $P_0 = \left[\sum_{n=0}^{s-1} 1/n! \left(\lambda/\mu\right)^n + \sum_{n=s} \frac{1}{s^{n-s}s!}\left(\lambda/\mu\right)^n\right]^{-1}$

5. $L_q = \frac{(s\rho)^s.\rho}{s!(1-\rho)^2}\left[1-\rho^{N-s+1} - (1-\rho)(N-s+1)\rho^{N-s}\right]P_0$

6. $L_s = L_q + s - P_0 \sum_{n=0}^{s-1} \frac{(s-n)(s\rho)^n}{n!}$

7. $W_s = \frac{L_s}{\lambda'}$; $\lambda' = \lambda(1-P_N)$

8. $W_q = W_s - 1/\mu$.

Unit - 4

Queueing Theory

Queuing system :-

A Queuing system is completly described by

1. Arrival Pattern (Input)
2. service pattern
3. Queue Discipline
4. Customer's behaviour

1. Arrival pattern (Input) (λ - poisson distribution)

Queuing system in which the customer arrive in paisson fashion . Mean arrival rate is denoted by (λ) and inter arrival rate is denoted by ($1/\lambda$)

2. Service pattern (μ - exponential distribution)

service time of each customers follows . exponential distribution . The main service rate is denoted by (μ) and inter service rate ($1/\mu$) .

3) Queue Discipline :-

Formation of the Queue, customer behaviour while rating and the manner of chosen service , the discipline are divided into three categories they are

1) FCFS (FIFO)
2) LCFS (LIFO)
3) SIRO

4) Customers behaviour :

Depends on customer divided into four categories .

i) Balking

Due to the no space, leaving Queue .

ii) Renging

Due to leave impatience .

iii) priorities

Refering priorities like quotes .

iv) Jockeying

jumping from one to another Queue

2m kendall notation for representative queuing models .

symbol from is (a/b/c) : (d/e)

a → arrival pattern

b → service pattern

c → No. of servers

d → No. of customers

e → Queue discipline

There are four methods of queuing models

Model I : (M/M/1) : (∞/FCFS)

μ - mean arrival

Model II : (M/M/S) : (∞/FCFS)

Model III : (M/M/1) : (N/FCFS)

Model IV : (M/M/S) : (N/FCFS)

Transient and steady state :

The system is said to be transient state when its operating characteristic are dependent on time.

A system is said to be in steady state when behaviour of system in dependent of time.

Model I :- $(M/M/I).(\infty/FCFS)$

① Average no. of units in the system

$L_s = \frac{\rho}{1-\rho}$ where $\rho = \lambda/\mu < 1$ is called density of the Queue.

λ - arrival rate

μ - service rate.

② Avg length of the Queue $L_q = L_s - \lambda/\mu$

or

$\frac{\rho^2}{1-\rho}$

③ Expected waiting time in the system,

$W_s = \frac{L_s}{\lambda}$ or $\frac{1}{\mu-\lambda}$

④ waiting time in the Queue, $W_q = L_q/\lambda$ or

$\frac{\lambda}{\mu(\mu-\lambda)}$

⑤ $p(\text{Queue size} \geq N) = \rho^N$

⑥ Expected length of non-empty Queue

$$(L>0) = \frac{\mu}{\mu-\lambda}$$

2nd// Little formula [Relation b/w Ls, Lq, Ws, Wq]

1. $Ls = \lambda Ws$

2. $Ws = \frac{1}{\mu-\lambda}$

3. $Lq = Ls - \frac{\lambda}{\mu}$

4. $Wq = Ws - \frac{1}{\mu}$

Q// In a railway marshaling yard, goods train arrive at a rate of 30 trains per day. Assuming that inter arrival time follows an exponential distribution. An service time is also exponential with the average of 36 min, calculate the following

i) The mean q size (Line length)

ii) The probability that q size exceeds 10.

iii) If the input of the train increases to an average 33 per day, what will be the changes in (i), (ii)?

soln: Arrival rate, $\lambda = \frac{30}{24 \times 60} = \frac{1}{48}$ min

Service rate, $\mu = \frac{1}{36}$ min

$$\rho = \frac{\lambda}{\mu} = \frac{\frac{1}{48}}{\frac{1}{36}} = 0.75 \text{ trains}$$

i) $L_s = \frac{\rho}{1-\rho} = \frac{0.75}{0.25} = 3$ trains

ii) $P(N \geq 10) = \rho^N = (0.75)^{10} = 0.05631$

iii) when the input increases to 33 trains/day

$$\lambda = \frac{33}{60 \times 24} = \frac{11}{480}$$

and $\mu = \frac{1}{36}$ hrs / min

$$\rho = \frac{\lambda}{\mu} = \left(\frac{11}{480}\right)(36)$$

$$\boxed{\rho = 0.825}$$

Now, $L_s = \frac{\rho}{1-\rho} = \frac{0.825}{1-0.825} = 4.714 \simeq 5$ trains

And also $P(N \geq 10)$

$$\rho^N = (0.825)^{10} = 0.146.$$

② In a supermarket, the average arrival rate of a customer is 10 in every 30 min following paison process. The average time taken by the cashier to and calculate the customer purchase is 2.5 min following exponential distribution. what is the probabity that the Queue length exceeds 6 and waiting time in the supermarket?

soln : $\lambda = \frac{10}{30}$ per min ; $\mu = \frac{1}{2.5}$ per min

$$\rho = \frac{\lambda}{\mu} = \frac{10}{30}(2.5)$$

$$= 0.8333$$

i) $P(\text{queue size} \geq N) = \rho^N$

ie, $P(\text{queue size} \geq 6) = (0.8333)^6$

$$= 0.3348$$

ii) Ws (waiting time) => $W_s = \frac{1}{\mu - \lambda}$

$$= \frac{1}{\left(\frac{1}{2.5}\right) - \left(\frac{10}{30}\right)} = 14.96 \text{ min}$$

Note :-

* 1. Probability of 'n' units in the system , $\boxed{P_n = \rho^n(1-\rho)}$; $\rho = \lambda/\mu$

2. $P(\text{waiting time in the system} \geq t) = \int_t^\infty (\mu - \lambda)e^{-(\mu-\lambda)t} d$

3. Customers arrive at one window drive - in bank accor to poison distribution with mean 10 per hr. Service time per customer is exponential with me 5 mins. The space in front of the window including that serviced cars can accommodate maximum of three cars. Other can wait outside the space.

i) What is the probability that the

arriving customer can drive directly to the space infront of the window.

(ii) What is the probability that an arriving customer can wait outside the indicated space.

(iii) How long the arriving customer is expected to wait before starting service?

Soln :-

$\lambda = 10$ per hr ; $\mu = 1/5 \times 60 = 12$ per hr

$\therefore$ density , $\rho = \lambda / \mu = \frac{10}{12} = 0.8333.$

p (n units in the system)

$$\boxed{P_n = \rho^n (1-\rho)}$$

i) prob that an arriving customer can drive directly to the space infront of the window

$= P_0 + P_1 + P_2$

where $P_0 = (1-\rho)$

$P_1 = \rho(1-\rho)$

$P_2 = \rho^2(1-\rho)$

$P_0 + P_1 + P_2 = (1-\rho)(1+\rho+\rho^2)$

$= (1-0.8333)(1+0.8333+(0.8333)^2)$

$= 0.421$

ii) $= 1-(P_0+P_1+P_2)$

$= 1-0.421$

$= 0.579.$

iii) Avg. waiting time of a custome in the queue.

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{10}{12(2)} = 0.417 \text{ hrs}$$

4. Arrivals at telephone booth are considered to be paisson with an average time of 10 mins b/w one arrival. The duration of phone call is assumed to be exponentially distributed in mean 3 min.

i) what is the probability that the person arriving at a booth will have to wait?

ii) A telephone department will install Second booth when convinced that an arrival expect waiting for at least 3 min for phone. By how much should the flow of arrivals increas in order to justify second bvth?

iii) Find the Average no. of units in the system. LS.

iv) Estimate fraction of the day that the phon will be inuse.

v) what is the probability that will take more than 10 mins all together to wait for phone and complete the call?

Soln:- $\lambda = \frac{1}{10} = 0.10$ min, $\mu = 1/3 = 0.33$ min.

i) $P(w>0) = \rho = \lambda/\mu$

$= \frac{\cancel{30}}{\cancel{10}}$ 0.3

$\boxed{\rho = 0.3}$

ii) Installation of 2nd booth:

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)}$$

ie, $3 = \frac{\lambda'}{0.33(0.33-\lambda')}$

where λ' is the arrival rate after installing 2nd booth.

$$3 = \frac{\lambda'}{0.1089 - 0.33\lambda'}$$

$$3(0.1089 - 0.33\lambda') = \lambda'$$

$$0.3267 - 0.99\lambda' = \lambda'$$

$$0.3267 = \lambda' + 0.99\lambda'$$

$$\lambda' = \frac{0.3267}{1.99}$$

$$\boxed{\lambda' = 0.164}$$

$\Rightarrow \lambda' - \lambda = 0.16 - 0.10$

$= 0.06$ arrivals /min increases.

iii) $L_s = \lambda W_s$

$$W_s = \frac{1}{\mu - \lambda}$$

$$= \frac{1}{0.33 - 0.10} = 4.34$$

$$L_s = 0.10 \times 4.34$$

$L_s = 0.43$ customers

(or)

$$L_s = \frac{\rho}{1-\rho} = \frac{0.3}{1-0.3} = 0.43 .$$

iv) Refer (i)

v) $p(W \geq 10)$

$$= \int_{10}^{\infty} (\mu - \lambda) \cdot e^{-(\mu-\lambda)t} dt$$

$$= \int_{10}^{\infty} (0.23)\, e^{-(0.23)t} dt$$

$$= \int_{10}^{\infty} 0.23\, e^{-(0.23)t} dt .$$

$$= \cancel{(0.23)} \left(\frac{e^{-(0.23)t}}{\cancel{0.23}} \right)_{10}^{\infty}$$

$$= -\left(e^{(-0.23)\infty} - e^{-(0.23)}\right.$$

$$= e^{-2.3} = 0.100$$

model II : $(M/M/s):(\infty/FCFS)$

1. Length of the queue $L_q = \frac{(\lambda/\mu)^s P_0}{s!} \frac{\rho}{(1-\rho)^2}$.

where $\rho = \frac{\lambda}{s\mu}$ and

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!\,(1-\rho)} \right]^{-1}$$

2. Length of the system $L_s = \lambda/\mu + L_q$.

3. Waiting time in queue

$$W_q = \frac{L_q}{\lambda} \text{ (or) } \frac{1}{\lambda} \cdot \frac{\rho(s\rho)^s}{s!\,(1-\rho)^s} \cdot P_0 .$$

4. Waiting time in the system

$W_s = L_s/\lambda$.

5. The mean waiting time in the Queue for those who actually wait is given by $(W/W>0) = \frac{1}{s\mu-\lambda}$.

Exp. length of the waiting time.

6. Probability of waiting time of the customer for service $\Big\}$ $P(W>0)$

$$= \frac{\left(\frac{\lambda}{\mu}\right)^s P_0}{s!(1-\rho)} \; ; \; P_s = \frac{(\lambda/\mu)^s P_0}{s!}$$

where $\rho = \frac{\lambda}{s\mu}$

1. A Telephone exchange has a long distance oper
Telephone company finds that during the peak hou
from a long distance call arrive in a poi
fashion on average rate of 15 per hr. The
length of service on this calls is approxim
exponentially distributed with mean leng
5 mins.

a) What is the probability that the Subscri
will have to wait for long distance call
during peak hour of a day.

b) If the subscribers will wait and are s
inturn, what is the expected waiting tim

soln:

Given,

$s = 2$;

$\lambda = \frac{15}{60} = \frac{1}{4}$ min

$\mu = \frac{1}{5}$ mins

$\rho = \frac{\lambda}{s\mu}$

$= \frac{\frac{1}{4}}{2 \times \frac{1}{5}} = 0.625.$

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!\,(1-\rho)}\right]^{-1}$$

$$= \left[\sum_{n=0}^{1} \frac{(2\cdot 5/8)^n}{n!} + \frac{(\cancel{2}\cdot 5/\overset{4}{\cancel{8}})^2}{2!\,\underset{0.375}{(\cancel{1-0.625})}}\right]^{-1}$$

$$= \frac{1}{1+(2.08)+5/4+(2.08)}$$

$$= 0.156 .$$

a) $$p(w>0) = \frac{(\lambda/\mu)^s P_0}{s!\,(1-\rho)}$$

$$= \frac{1.5625 \times 0.156}{2!\,(0.375)}$$

$$= \frac{0.24375}{0.75}$$

$$= 0.325 .$$

b) $$W_q = L_q/\lambda \text{ (or) } \frac{1}{\lambda} \cdot \frac{\rho(s\rho)^s}{s!\,(1-\rho)^2} \cdot P_0$$

$$= \frac{\cancel{4}\cdot\left(\frac{5}{\cancel{8}_2}\right)\left(\frac{5}{4}\right)^2}{2!\,(1-5/8)^2} \cdot 0.156 .$$

$$= \frac{3.90625 \times 0.156}{0.28125}$$

$$= 2.166 \simeq 2.167, \text{ min} .$$

2. A supermarket has 2 girls ringing up sa
at the counters if the service time for
each customer is exponential with mean 4 m
If the people arrive in the possion Fashio
10 per hr.

a) What is the probability of having to wait for service ?

b) What is the expected % of idle time for each girl ?

c) If a customer has to wait, what is the expected length of the waiting time ?
[conditional probability]

Soln :-

$$s = 2 \; ; \; \lambda = \frac{1\cancel{0}}{6\cancel{0}} = \frac{1}{6} \text{ min} \; ;$$

$$\mu = \frac{1}{4} \text{ min} .$$

$$\rho = \frac{\lambda}{s\mu} = \frac{\frac{1}{6}}{\cancel{2} \times \frac{1}{\cancel{4}_{2}}}$$

$$\rho = 0.33$$

a) $p(w>0)$ . .

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!(1-\rho)} \right]^{-1}$$

$$P_0 = \left[\sum_{n=0}^{1} \frac{(2 \times 1/3)^n}{n!} + \frac{(2 \times 1/3)^2}{2!(1-0.33)} \right]^{-1}$$

$$= \frac{1}{1 + (0.332) + (2/3)}$$

$$= \frac{1}{1 + (0.332) + 0.66}$$

$$= \frac{1}{1.992} = 0.5$$

$$P(W>0) = \frac{(\lambda/\mu)^s P_0}{s!(1-\rho)}$$

$$= \frac{(4/6)^2 (0.5)}{2!(1-0.33)}$$

$$= 0.166 .$$

b) The fraction of the time, the service is busy.

$$\rho = \frac{\lambda}{s\mu} = \frac{1}{3} = 0.333$$

$$\therefore \left.\begin{array}{l}\text{The fraction of the time} \\ \text{Service remains idle}\end{array}\right\} = 1 - \rho$$

$$= 1 - 0.333$$

$$= 0.667$$

$$= 67\%$$

(approx).

c) $P(W/W>0) = \frac{1}{s\mu - \lambda}$

$$= \frac{1}{(\cancel{2}\cdot 1/\cancel{4}_2 - 1/6)}$$

$$= \frac{1}{\frac{3-1}{6}}$$

$$= \cancel{6}/\cancel{2}^{3} \simeq 3 \text{ min}$$

3. A petrol station has two pumps. The service ti
follows the exponential distribution with mean
4 min and cost of arrival time for service in
poisson process at a rate 10 custs per hr. Find
the probability that the customer has to wait fo
service. what proportion of time the pump
remains idle ?

soln: $s = 2$; $\lambda = 10\,hr$; $\mu = \frac{1}{4}$ min

$= 15$ hr.

$$\rho = \frac{\lambda}{s\mu} = \frac{\cancel{10}\,5}{\cancel{2}\times\cancel{15}_{3}}$$

$$\boxed{\rho = 0.333}$$

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^2}{s!(1-\rho)}\right]^{-1}$$

$$= \left[\sum_{n=0}^{1} \frac{(2 \times 0.33)^n}{n!} + \frac{(2 \times 0.33)^2}{2!\, 0.667} \right]^{-1}$$

$$= \left[\frac{1}{1 + 0.666 + 0.335} \right]$$

$$= \frac{1}{2.002}$$

$$\boxed{P_0 = 0.499} \simeq 0.5$$

a) $P(W>0) = \frac{(\lambda/\mu)^s P_0}{s!\,(1-\rho)}$

$$= \frac{\left(\frac{\cancel{10}^{2}}{\cancel{15}_{3}}\right)^2 \cdot 0.5}{2!\; 0.667}$$

$$= \frac{0.222}{1.334}$$

$$= 0.167.$$

b) The fraction of time, the service is busy.

$$\rho = \frac{\lambda}{s\mu} = \frac{10}{15 \times 2} = 1/3 = 0.333$$

The fraction of time the remains idle $\Big\} = 1 - \rho$

$$= 1 - 0.333$$

$$= 0.667\,\%.$$

$$= 67\,\%.$$

(approx)

Model III : (M/M/1) : (N / FCFS)

1. The probability of the system is empty $P_0 = \frac{1-}{1-}$

   where $\rho = \lambda/\mu$ No. of custo

2. The system with 'n' no. of customers

$$P_n = \frac{1-\rho}{1-\rho^{N+1}} \cdot \rho^{n} \; ; n = 0 \text{ to } N.$$

3. Length of the system, $L_s = P_0 \sum_{n=0}^{N} n \rho^n$

4. Length of the Queue, $L_q = L_s - \lambda/\mu$

5. Waiting time in the system $W_s = L_s/\lambda$

6. Waiting time in the Queue, $W_q = L_q/\lambda$

1. If for a period of 2 hours in a day (8 - 10 AM) trains arrive the yard every 20 mins, But service time continous to remain 36 mins. Then calculate for this period.

   a) The probability that the yard is empty

   b) Average Queue length, assuming that capa of yard is 4 trains only.

soln : $N = 4$; $\lambda = \frac{1}{20}$ min ; $\mu = \frac{1}{36}$ min

$$\rho = \lambda/\mu = \frac{1}{20} \times \frac{36}{1}$$

$$\boxed{\rho = 1.8}$$

a) The probability of yard is empty.

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-1.8}{1-(1.8)^5}$$

$$= \frac{\cancel{-}0.8}{\cancel{-}17.8}$$

$$P_0 = 0.04$$

b) Average queue length, (or) length of system

$$L_s = P_0 \sum_{n=0}^{N} \rho^n$$

$$= 0.04\,[0 + 1.8 + 6.48 + 17.49 + 41.99]$$

$$= 0.04\,[67.76]$$

$$= 2.7 \simeq 3 \text{ trains}$$

2. A Barbar shop has space to accomodate only 10 customers. It can service only one person at a time. If a customer comes to his shop and find its full, to goes to the next shop and customers randmly arrive at an average rate $\lambda = 10$ per hour and the barbar's

service time is negative Exponential with an average of $1/\mu = 5$ min per customer. Find a shop is empty, and the shop with 5 customer

soln : $N = 10$; $\lambda = 10$ hrs $\qquad \mu = 1/5$ min

$= \frac{10}{60} = 1/6$ min

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}}$$

$$\rho = \lambda/\mu = \frac{5}{6} = 0.83$$

$$\boxed{\rho = 0.83}$$

$$P_0 = \frac{1-0.83}{1-(0.83)^{11}} = \frac{0.17}{0.87}$$

$$\boxed{P_0 = 0.195}$$

$$P_5 = \left(\frac{1-\rho}{1-\rho^{N+1}}\right)\rho^{n..}$$

$$= (0.195)\,(0.83)^5$$

$$= 0.195 \times 0.393$$

$$\boxed{P_5 = 0.076}$$

125

3. A car park contains 5 cars the arrival of cars is poisson at a mean rate of 10 per hr. The length of time each car spends in the car park is negative exponential distribution with mean of 2 hr. How many cars are in the car park of average.

soln:

$N = 5, \quad \lambda = \frac{10}{60} = \frac{1}{6}$ min

$\mu = \frac{2}{2 \times 60} = \frac{1}{120}$ min

$\rho = \lambda / \mu$

$= \frac{1}{6} \times \frac{120}{1}$ 20

$$\boxed{\rho = 20}$$

$$P_0 = \frac{1 - \rho}{1 - \rho^{N+1}}$$

$$= \frac{1 - 20}{1 - (20)^6}$$

$= 2.968 \times 10^{-7}$

$= 0.0000002962$

The cars in the car park averagely
Avg. length

$$L_s = P_0 \sum_{n=0}^{N} n \rho^n$$

$$L_s = P_0 \sum_{n=0}^{N} n e^n$$

$$= (2.962 \times 10^{-7}) \sum_{n=0}^{5} n(20)^n$$

$$= (2.962 \times 10^{-7}) [0 + 20 + 2 \times 400 + 3 \times (20)^3 + 4 \times (20)^4 + 5 \times (20)^5].$$

$$= (2.962 \times 10^{-7})$$

$$= 4.9 \simeq 5 \text{ approx.}$$

4. In a Railway marshaling yard where the goods train arrive at the rate of 30 trains per day. Assume that interval arrival tim follows an exponential distribution at the service time is also to be assumed as exponential with mean of 36 min. calculat

a) The probability that the yard is empty.

b) The average Queue length assuming tha line capacity of the yard is 9 trains.

soln: $N = 9, \quad \lambda = \frac{\cancel{30}}{24 \times \cancel{60}_{2}} = \frac{1}{48}$ min

$$\mu = \frac{1}{36} \text{ min}$$

$$e = \frac{36}{48} = \frac{2}{3}$$

$$\boxed{e = 0.75}$$

a) $P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-0.75}{1-(0.75)^{10}}$

$= \frac{0.25}{0.94}$

$\boxed{P_0 = 0.265}$

b) Avg. Queue Length

$L_s = P_0 \sum_{n=0}^{N} n\rho^n$

$= (0.265) \sum_{n=0}^{9} n(0.75)^n$

$= 0.265 [0 + 0.75 + 2(0.75)^2 + 3(0.75)^3 + 4\times(0.75)^4 + 5(0.75)^5 + 6(0.75)^6 + 7(0.75)^7 + 8(0.75)^8 + 9(0.75)^9]$

$= 0.265 \times [0.75 + 1.125 + 1.265 + 1.265 + 1.186 + 1.067 + 0.934 + 0.800 + 0.675]$

$= 0.265 \times 9.067$

$\boxed{L_s = 2.4}$

0.75
1.125
1.265
1.265
1.186
1.067
0.934
0.800
0.675

9.067

Model - IV :

1. A barbar shop has 2 barbars and 3 chairs fo customers. Assume that the customers arrive in poisson fashion at a rate of 5 per hour. Eac barbar services customers according to an exponential distribution with mean 15 mins. Further if a customer arrives and there are no empty chairs in the shop, he will leave what is the expected no of customers in the shop?

Soln:

$S = 2, \ N = 3, \ \lambda = \frac{5}{60} = \frac{1}{12} \text{ min} ; \ \mu = \frac{1}{15} \text{ min}$

$\lambda / \mu = \frac{15}{12} = 5/4$

$\rho = \lambda / s\mu = \frac{5}{2 \times 4} = \frac{5}{8} = 0.625.$

$P_0 = \left[\sum_{n=0}^{1} \frac{1}{n!} (\lambda/\mu)^n + \sum_{n=2}^{3} \frac{1}{2^{n-2}} \cdot \frac{1}{2!} (\right.$

$= \left[\left(1 + \left(\frac{5}{4} \right) \right) + \left(\frac{1}{2!} \left(\frac{5}{4} \right)^2 + \frac{1}{2.2!} \left(\frac{5}{4} \right)^3 \right] \right.$

$P_0 = 0.28$

and $L_q = \frac{[2 \times (0.625)]^2 (0.625)}{2! (1 - 0.625)^2} [1 - (0.625$

$(1 - 0.625) ($

$(0.625)]$

$(0.2$

$= 0.138$

$$L_s = L_q + s - P_0 \sum_{n=0}^{s-1} \frac{(s-n)(sP)^n}{n!} .$$

$$= 0.138 + (2) - 0.28 \left[(2) + 2(0.625) \right]$$

$$=$$

2m

Simulation :

Simulation is a representation of a system model that has designed characteristics of reality in order to produce the actual operation.

Ex : Testing and haircraft model.
planetorium

NOTE :

Monte-carlo method is a simulation technique in which statistical distribution functions are created by using a series of random numbers. By Monte-carlo method we can obtain the solution of nearest approach [Not optimal]

1. An automobile production line turns out about 100 cars a day. But deviation occurs owing to many causes. The production is more accurately described by the probability distribution Given below.

Production/day	Probability	Prod/day	Probabil
95	0.0.3	101	0.15
96	0.05	102	0.10
97	0.0.7	103	0.0
98	0.10	104	0.05
99	0.15	105	0.03
100	0.20		

Finished cars are transported across the bay. If the ferry has space for only 101 cars, what will be the average no. of cars waiting to be shipped. And what will be the average no. of empty space on the ship.

soln: The tag-numbers are established in the table below.

Prod/day	Prob	Cummulative	Tag-Num
95	0.03	0.03	00 - 02
96	0.05	0.08	03 - 07
97	0.07	0.15	08 - 14
98	0.10	0.25	15 - 24
99	0.15	0.40	25 - 39
100	0.20	0.60	40 - 59
101	0.15	0.75	60 - 74
102	0.10	0.85	75 - 84

103	0.07	0.92	85-91
104	0.05	0.97	92-96
105	0.03	1.00	97-99

The simulated production of cars for the next 15 days is given in the following table.

Day	Random number	Prod/day	No. of. cars	No. of. empty space.
1.	97	105	4	–
2.	02	95	~~6~~	6
3.	80	102	1	–
4.	66	101	–	–
5.	96	104	3	–
6.	55	100	–	1
7.	50	100	–	1
8.	29	99	–	2
9.	58	100	–	1
10.	51	100	–	1
11.	04	96	–	5
12.	86	103	2	–
13.	24	98	–	3
14.	39	99	–	2
15.	47	100	–	1
		Total	10	23

i) Avg no. of cars waiting to be shipped

$$= \frac{10}{15} = 0.67 \text{ per day.}$$

ii) Avg. no. of empty spaces on the ship $= \frac{23}{15}$
$= 1.53$ per day.

Poisson distribution :

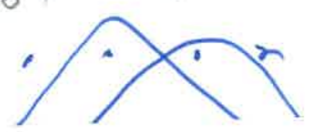
Q11 Mean = 5, then generate 20 days of sales by Monte Carlo method. The probability for sales

$$p(X=r) = \frac{e^{-\lambda}.\lambda^{x}}{\cancel{x}!} \qquad \lambda = 0,1,2,\dots \infty.$$

$$p(x=5) = \frac{e^{-5}.5^{r}}{x!} \qquad [\lambda = \text{mean} = 5]$$

x	probability	cumulative property	Tag number
0	0.006	0.006 ≃ 0.01	00 - 00
1	0.03	0.036 ≃ 0.03	01 - 03
2.	0.08	0.116 ≃ 0.12	04 - 11
3.	0.14	0.256 ≃ 0.3	12 - 29
4.	0.17	0.426 ≃ 0.4	30 - 39
5.	0.17	0.596 ≃ 0.6	40 - 59
6.	0.14	0.742 ≃ 0.7	60 - 69
7.	0.10	0.842 ≃ 0.8	70 - 79
8.	0.06	0.902 ≃ 0.9	80 - 89

9.	0.03	0.932 ≃ 0.9	90-90
10.	0.018	0.95 ≃ 0.95	91-94
11.	0.008	0.958 ≃ 0.96	95-95
12.	0.003	0.961 ≃ 0	95-95



4/11/22

Unit - V

Inventory Models

Inventory may be defined as the stock of goods, commodities, or other economic resources that are stored and received for smooth and efficiently running of business affairs.

Reasoning for maintaining Inventories:

1. It helps in smooth and efficient runni of the bussiness.

2. It provides adequate service to the Custon

3. It reduces the possibility of duplication of orders.

4. It acts as a buffer stocks when raw materials are received late and shop rejection are too many.

5. Takes Advantages of price and Discounts by bulk purchasing.

Inventory costs:

There are four categories of Inventory costs.

1. Item (production or purchase) cost.

2. Carrying or Holding cost (C_1)

3. Shortage cost (C2) :

4. Ordering (or) setup cost (C3) :

Item cost :

Cost associated with an item whether it is manufactured or purchased.

C1 :

Cost of holding the unit of stock in a inventory in unit time, is known as holding cost.

It includes storage expensives, interest on capital, insurance and taxes, security, damage etc...

Shortage cost or stock out cost (C2) :

The penalcity cost that are included as a result out of stock.

Ordering or setup cost (C(3)) :

Its associated with optaining goods through purchasing, manufacturing before starting production.

Economic ordered quantity (EOQ) :

As the quantity ordered [lot size 'q'] Carrying cost decreases, but ordering cost (C3) increases.

∴ The economic ordered quantity (EOQ)

is that the size of order which minimize th
total of carrying / ordering cost of inventory.

Types of Inventory models :

1) Deterministic Inventory model.

2) Probabilistic Inventory model.

Deterministic Inventory model (Economic lot size
model)

There are 4 models :

MODEL I : Purchasing model with no shortag
" II : Manufacturing model with no
shortag

MODEL III : Purchasing model with shortages.
" IV : Manufacturing model with shorta

MODEL I : Purchasing model with no shortage

The assumptions for this model is as follow

i) Demand rate is uniform (R)

ii) Production rate is infinite. (K)

iii) shortages are not allowed.

iv) Lead time is known exactly or '0'.

[The goods available to uses].

Formulae :

1. Optimum no of orders placed per year, $N = \frac{R}{q_0}$

$$= \sqrt{\frac{RC_1}{2C_3}}$$

R - uniform demand rate

$q_0 (q^*)$ - Economic order quantity (EOQ).

C_1 - Holding cost

C_3 - ordering cost .

2. optimum length of time b/w consecutive orders

$$t_0 = \frac{q_0}{R} = \sqrt{\frac{2C_3}{RC_1}}$$

3. Minimum total Amount inventory cost,

$$C_0 = \sqrt{2RC_1C_3}$$

4. EOQ (optimum lot size)

$$q_0 = \sqrt{\frac{2RC_3}{C_1}}$$

5. Average Inventory carrying cost $= 1/2 q_0 C_1$.

6. Total ordering cost $= R/q_0 C_3$.

7. Total annual cost of the existing inventory policy

8. Optimum cost $= (R \times \text{price/unit}) + \sqrt{2C_1C_3R}$

Q. The annual demand of item is 3200 units. The unit cost is rupees 6/- and inventory carrying cost 25% per annum. If the cost of procurement is Rs 150/-

i) Find EOQ

ii) No. of. orders per year

iii) Time b/w two consecutive orders.

iv) optimum cost.

Soln given

Demand (R) = 3200

carrying cost $(C_1) = C \times I = \overset{3}{\cancel{6}} \times \frac{25}{\cancel{100}_{2}} = 1$

ordering cost $(C_3) = 150/-$

i) EOQ.

$$q_0 = \sqrt{\frac{2 \times 3200 \times 150}{1.5}} = 800 \text{ units}$$

ii) $N = \frac{R}{q_0} = \frac{3200}{800} = 4$

iii) Time b/w two consecutive orders.

$$t_0 = \frac{q_0}{R}$$

$$= \frac{800}{3200} = \frac{1}{4} = 0.25 \text{ years.}$$

[3 months].

iv) optimum cost.

$$= (R \times \text{price/unit}) + \sqrt{2 C_1 C_3 R}$$

$$= 192000 + 1200$$

$$= \text{Rs. } 204000/-$$

2. The demand rate of particular is 12,000 units. The set up is Rs. 350 per annum.

soln: Given, $R = 12{,}000$ per year;

$C_1 = 0.20$ per month;

$C_3 =$ Rs. 350 per annum;

$= 0.20 \times 12$

$=$ Rs. 2.4 per year.

i) optimum run size, $q_0 = \sqrt{\frac{2RC_3}{C_1}}$

$$= \sqrt{\frac{2(1200)(350)}{(2.4)}}$$

$$= \text{Rs. } 1870.82$$

ii) optimum scheduling period, $t_0 = q_0/R$.

$$= \frac{1871}{12,000} \text{ per year}$$

$$= \frac{1871 \times 12}{12,000} \text{ months}$$

$$= 1.871 = 1.9 \text{ months}.$$

iii) Tot inventory cost, $C_0 = \sqrt{2R C_1 C_3}$

$$= \sqrt{2 \times 1200 \times 2.4 \times 350}.$$

$$= 4489.9$$

$$\simeq 4490/-$$

3. The annual requirement for a model is 3000 units. The ordering cost is ₹ 100 per order. The cost per unit is ₹ 10. Carrying cost per year is 30% of the unit cost.

a) Find EOQ.

b) By using better organisational methods

The ordering cost per order is brought down to ₹ 80 per order. But the same quantity has determined above were ordered.

c) If a new EOQ is found by using the ordering cost has 80 ₹, what would be further savings in cost?

Soln: Given $R = 3000$ units / year, $C_1 = C \times I$

$$= 10 \times \frac{30}{100} = 3 \text{ per unit/yr.}$$

$C_3 =$ Rs. 100 per order.

a) optimal lot size (EOQ), $q_0 = \sqrt{\frac{2RC_3}{C_1}}$

$$= \sqrt{\frac{2(3000)(100)}{3}}$$

$= 447$ units.

Total inventory cost $= \sqrt{2RC_1C_3}$

$$= \sqrt{2 \times 3000 \times 3 \times 100}$$

$=$ ₹ 1342

b) $R = 3000$ unit/per, $C_1 = 3$ per unit/yr

$C_3 =$ Rs. 80 per order.

optimal lot size $(q_0) = \sqrt{\frac{2RC_3}{C_1}}$

$= \sqrt{\frac{2 \times 3000 \times 80}{3}} = 400$ units/order.

Total inventory cost $= \frac{R}{q_0} C_3 + \frac{q_0}{2} C_1$

$= \frac{3000}{\cancel{400}} \times 100 + \frac{\overset{2}{\cancel{400}}}{\cancel{2}} \times 3$

$= \frac{30 \times \overset{25}{\cancel{100}}}{\cancel{4}} + 200 \times 3$

$= 30 \times 25 + 600$

$= Rs. 1350.$

Net change in the total cost (or) $= 1350 - 1342$

Saving in cost $= Rs. 8 \; (0.08)\%.$

Model II : Manufacturing model with no shortages

{∴ production rate $k >$ Demand rate R}

Characteristics :-

i) EOQ (optimum lot size), $q_0 = \sqrt{\frac{2RC_3}{C_1}} \cdot$

ii) optimum length of the production run, $t_0 = \frac{q_0}{R}.$

iii) optimum no of production run/year, $N = \frac{R}{q_0}$

iv) Tot. Minimum cost, $C_0 = \sqrt{2RC_1C_3} \sqrt{\frac{K-R}{K}}.$

v) Maximum inventory $= \frac{q_0}{k} (K-R)$

vi) Time of manufacture $= q_0/k.$

1. The Demand Rate of an item in a Company is 18,000 units per year. The company can produce at a rate of 3000 per month. The set up cost is rupees 500 per order. The holding cost per unit/per month is ₹0.15. Calculate

(i) optimum manufacturing quantity
ii) Maximum Inventory
iii)
iv) No. of order per year
v) Time of manufacture
vi) optimum annual cost if the cost of item is ₹ 2 per unit.

Soln: Demand Rate, R $= Rs. \frac{18000}{12}$ per unit/yr (M→Y÷12)
Y→M×12)

Prod. rate $k = 3000$ units/Month.

$C_1 = Rs. 0.15$ units/Month

$C_3 = Rs. 500$/order.

i) optimum quanity, $q_0 = \sqrt{\frac{2RC_3}{C_1}} \cdot \sqrt{\frac{k}{k-R}}$

$$= \sqrt{\frac{2 \times \frac{18000}{12} \times 500}{0.15}} \cdot \sqrt{\frac{3000}{3000-150}}$$

= 4471 units

10,0000
3162.27

ii) Max. inventory $= \frac{q_0}{K}(K-R)$

$= \frac{4471}{3000}(3000-1500)$

$= 2236.$

iii) Time b/w order, $t_0 = \frac{q_0}{R}$

$= \frac{4471}{1500} = 2.98$

iv) The no. of order/year $N = R/q_0 = \frac{18000}{4471} = 4$

v) The time of manufacture $= q_0/K = \frac{4471}{3000}$

$= 1.49$

vi) The optimum annual cost $= (R \times C) + \sqrt{2RC_1C_3}\sqrt{}$

$= (2 \times 18000) + \sqrt{2(1800)(0.15)(500)}\sqrt{\frac{3000-150}{3000}}$

=

2. A company has a demand of 12,000 units per year for an item and it can produce 2000 items per month. The setup cost ₹ 400 & holding cost per unit per month is 1.05. Find the optimum lot size, Max inventory, manufacturing time.

q0 $\frac{q_0}{K}(K-R)$ q_0/K

Soln:

$R = 12000$ units/yr.

$K = 2000$ units/Month

$= 24000$ per year.

$C_3 = ₹\ 400$

$C_1 = ₹\ 0.15$ per unit/month

$= ₹\ 1.8$ per unit/yr.

i) optimum lot size (EOQ) $q_0 = \sqrt{\frac{2RC_3}{C_1}} \sqrt{\frac{K}{K-R}}$

$$= \sqrt{\frac{2(12000)(400)}{1.8}} \sqrt{\frac{24000}{24000-12000}}$$

$= 3265.5 \simeq 3266$ units

ii) Max. inventory $= \frac{q_0}{K}(K-R)$

$$\overset{1633.5}{=} \frac{\cancel{3266}_{20}}{\cancel{24000}_{2}} \times \cancel{12000}$$

$= 1633$ units

iii) Manufacturing time $= \frac{q_0}{K}$

$$= \frac{3266}{24000}$$

$= 0.136$ years.

$= 49.64$

$\simeq 50$ days.

iv) time $t_0 = \frac{q_0}{R} = \frac{3266}{12000} = 0.272$ yr

$\simeq 99.2$ days.

3) An item is produced at the rate of \$50 items per / day. The demand occurs at the rate of 25 item per day. If the set up cost is Rs: 100/ per / ~~Month~~ run. And holding cost 0.01 per unit of item per day. Find the economic lot size (run, assuming that the shortage are not permitted. Find the time of cycle and minimum cost for 1 run.

Soln :

$R = 25$ items / day

$K = 50$ items / day.

$C_3 = ₹\ 100/$ run

$C_1 = ₹\ 0.01$ per unit / day

i) optimum (lot size) $q_0 = \sqrt{\frac{2RC_3}{C_1}} \sqrt{\frac{K}{K-R}}$

$= \sqrt{\frac{2(25)(100)}{0.01}} \sqrt{\frac{50}{25}}$

$= 999.8$

$\simeq 1000$ units

ii) time of cycle $t_0 = \frac{q_0}{R}$

$= \frac{\overset{40}{\cancel{1000}}}{\cancel{25}} = 40$ days.

iii) Minimum cost per cycle, $c_0 = \sqrt{2RC_1C_3}\sqrt{\frac{k-R}{k}}$

$$= \sqrt{2 \times 25 \times 0.01 \times 100}\sqrt{\frac{50-25}{50}}$$

$$= \sqrt{50 \times 0.01 \times 100}$$

$$= 7.07 \times 0.707$$

$$= 4.89$$

$$\simeq ₹\ 5/-$$

Min cos per run = ₹ 5 × 40

= ₹200/-

Model iii : Purchasing model with shortages :-

1. EOQ, $q_0 = \sqrt{\frac{2C_3R}{C_1}\left(\frac{C_1+C_2}{C_2}\right)}$; C_2 = shortage cost.

2. period, $t_0 = q_0/R$

3. No. of orders, $N = R/q_0$

4. Minimum cost, $c_0 = \sqrt{2C_1C_3R}\sqrt{\frac{C_2}{C_1+C_2}}$

5. No. of strategies $= q_0\left(\frac{C_1}{C_1+C_2}\right)$

[max. backorder $= q_0 - q_0'$; $q_0' = \frac{C_2}{C_1+C_2}$]

6. Total cost the period

$$= C \times R + \sqrt{2RC_1C_3} \sqrt{\frac{C_2}{C_1+C_2}}$$

Q2) The demand of an item is 18,000 units per year. The holding cost is Rs. 1.20 per unit time. And the cost of shortage Rs. 5-/-

Soln:

$R = 18000$ units/yr. ; $C_1 = ₹1.20$ per unit

$C_2 = ₹5$; $C_3 = ₹400$

i) EOQ, $$q_0 = \sqrt{\frac{2C_3R}{C_1}\left(\frac{C_1+C_2}{C_2}\right)}$$

$$= \sqrt{\frac{2(400)(18000)}{(1.2)}\left(\frac{(1.2+5)}{5}\right)}$$

$= 3857$ units

ii) optimum period $t_0 = \frac{q_0}{R} = \frac{3857}{18000} = 0.2142$

iii) The Numbers of order per year $N = \frac{R}{q_0}$

$$= \frac{18000}{3857}$$

$$= 4.66$$

$$= 4.67.$$

2) The demand for an item is ₹12000 per/year and shortage ~~allowed~~ unknown. and ~~carrying~~ unit cost is 15 holding cost is ₹ 20 per year. Determine, the no of orders per year the optimum total yearly cost. The cost of placing 1 order is ₹ 6000. The cost of one shortage is ₹ 100 per year.

Sol:

$R = 12000$ per/year

$C_1 = ₹ 20$ per unit /year

$C_2 = ₹ 100$ /year

$C_3 = ₹ 6000$

$$EOQ,\ q_0 = \sqrt{\frac{2C_3R}{C_1}\left(\frac{C_1+C_2}{C_2}\right)}$$

$$= \sqrt{\frac{2(6000)(12000)}{20}\left(\frac{200+100}{100}\right)}$$

$= 2939$ units

i) No of orders/year, $N = \frac{R}{q_0} = \frac{12000}{2939} = 4.08$ order

ii) No of Shortages $= q_0 \times \left(\frac{C_1}{C_1+C_2}\right)$

$$= (2939)\left(\frac{20}{20+100}\right)$$

$= 489.83$

$\simeq 490$

iii) Total yearly cost $= C \times R + \sqrt{2RC_1C_3}\sqrt{\frac{C_2}{C_1+C_2}}$

$$= 15 \times 12000 + \sqrt{\frac{2(12000)(20)(6000)}{(20)(6000)}}\left(\sqrt{\frac{100}{20+100}}\right)$$

$= ₹ 228989$

③ Given following data from an unit of an item from demand instantaneously delivery tim & shortage. Annual demand = 800 unit, cost of an item = ₹ 40, order cost = ₹ 800, Inventory carrying cost = 40% shortage = ₹ stock out cost). Find

i) Min m order qty (EOQ)

ii) Max no. of backorder shortage

iii) Time b/w orders

iv) Total annual cost

$R = 800$ units

$C_3 = ₹800$

$C_1 = 40 \times \frac{40}{100}$

$\boxed{C_1 = 16 ₹}$

$C_2 = ₹ 10$.

i) $q_0 = \sqrt{\frac{2RC_3}{C_1}} \sqrt{\frac{C_1+C_2}{C_2}}$

ii) $q_0 \times \left(\frac{C_1}{C_1+C_2}\right)$.

iii) $t_0 = q_0/R$.

iv) Total annual cost $= C \times R + \sqrt{2RC_1C_3} \sqrt{\frac{C_2}{C_1+C_2}}$.