

# **JEPPIAAR ENGINEERING COLLEGE**

## **DEPARTMENT OF ECE**

### **VISION OF DEPARTMENT**

To become a centre of excellence to provide quality education and produce creative engineers in the field of Electronics and Communication Engineering to excel at international level.

### **MISSION OF DEPARTMENT**

|           |  |
|-----------|--|
| <b>M1</b> | Inculcate creative thinking and zeal for research to excel in teaching-learning process.   |
| <b>M2</b> | Create and disseminate technical knowledge in collaboration with industries.   |
| <b>M3</b> | Provide ethical and value based education by promoting activities for the betterment of the society.   |
| <b>M4</b> | Encourage higher studies, employability skills, entrepreneurship and research to produce efficient professionals thereby adding value to the nation's economy. |

### **PEO of DEPARTMENT**

|                |   |
|----------------|---|
| <b>PEO I</b>   | Produce technically competent graduates with a solid foundation in the field of Electronics and Communication Engineering with the ability to analyze, design, develop, and implement electronic systems. |
| <b>PEO II</b>  | Motivate the students for successful career choices in both public and private sectors by imparting professional development activities.  |
| <b>PEO III</b> | Inculcate in the students' ethical values, effective communication skills and develop the ability to integrate engineering skills to broader social needs.  |
| <b>PEO IV</b>  | Impart professional competence, desire for lifelong learning and leadership skills in the field of Electronics and Communication Engineering.   |

### **PSO of DEPARTMENT**

|                |   |
|----------------|---|
| <b>PSO I</b>   | Competence in using modern electronic tools in hardware and software co-design for networking and communication applications.                 |
| <b>PSO II</b>  | Promote excellence in professional career and higher education by gaining knowledge in the field of Electronics and Communication Engineering |
| <b>PSO III</b> | Understand social needs and environmental concerns with ethical responsibility to become a successful professional.                           |

## **SYLLABUS**

### **MA6451 PROBABILITY AND RANDOM PROCESSES**

#### **OBJECTIVES:**

To provide necessary basic concepts in probability and random processes for applications such as random signals, linear systems etc in communication engineering.

#### **UNIT I RANDOM VARIABLES**

Discrete and continuous random variables - Moments - Moment generating functions - Binomial, Poisson, Geometric, Uniform, Exponential, Gamma and Normal distributions.

#### **UNIT II TWO - DIMENSIONAL RANDOM VARIABLES**

Joint distributions - Marginal and conditional distributions - Covariance - Correlation and Linear regression - Transformation of random variables.

#### **UNIT III RANDOM PROCESSES**

Classification - Stationary process - Markov process - Poisson process - Random telegraph process.

#### **UNIT IV CORRELATION AND SPECTRAL DENSITIES**

Auto correlation functions - Cross correlation functions - Properties - Power spectral density - Cross spectral density - Properties.

#### **UNIT V LINEAR SYSTEMS WITH RANDOM INPUTS**

Linear time invariant system - System transfer function - Linear systems with random inputs - Auto correlation and Cross correlation functions of input and output.

#### **TEXT BOOKS:**

1. Ibe.O.C., "Fundamentals of Applied Probability and Random Processes", Elsevier, 1st Indian Reprint, 2007.
2. Peebles. P.Z., "Probability, Random Variables and Random Signal Principles", Tata Mc Graw Hill, 4th Edition, New Delhi, 2002.

#### **REFERENCES:**

1. Yates. R.D. and Goodman. D.J., "Probability and Stochastic Processes", 2nd Edition, Wiley India Pvt. Ltd., Bangalore, 2012.
2. Stark. H., and Woods. J.W., "Probability and Random Processes with Applications to Signal Processing", 3rd Edition, Pearson Education, Asia, 2002.
3. Miller. S.L. and Childers. D.G., "Probability and Random Processes with Applications to Signal Processing and Communications", Academic Press, 2004.
4. Hwei Hsu, "Schaum's Outline of Theory and Problems of Probability, Random Variables and Random Processes", Tata Mc Graw Hill Edition, New Delhi, 2004.
5. Cooper. G.R., Mc Gillem. C.D., "Probabilistic Methods of Signal and System Analysis", 3rd Indian Edition, Oxford University Press, New Delhi, 2012.



# JEPPIAAR ENGINEERING COLLEGE

## MA6451 - PROBABILITY AND RANDOM PROCESSES

### UNIT I RANDOM VARIABLES

#### PART A

1. If a random variable X takes values 1, 2, 3, 4 such that  $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$ . Find the probability distribution of X. (A/M 2015)

**Solution:**

Assume  $P[X = 3] = \alpha$ .

By the given equation  $P[X = 1] = \frac{\alpha}{2}, P[X = 3] = \frac{\alpha}{3}, P[X = 4] = \frac{\alpha}{5}$

For a probability distribution and mass function)  $\sum_x p(x) = 1$

$$p(1) + p(2) + p(3) + p(4) = 1$$

$$\frac{\alpha}{2} + \frac{\alpha}{3} + \alpha + \frac{\alpha}{5} = 1 \quad \Rightarrow \quad \frac{61}{30}\alpha = 1 \quad \Rightarrow \quad \alpha = \frac{30}{61}$$

$$P(X = 1) = \frac{15}{61}; P(X = 2) = \frac{10}{61}; P(X = 3) = \frac{30}{61}; P(X = 4) = \frac{6}{61}$$

The probability distribution is given by

|        |                 |                 |                 |                |
|--------|-----------------|-----------------|-----------------|----------------|
| X      | 1               | 2               | 3               | 4              |
| $p(x)$ | $\frac{15}{61}$ | $\frac{10}{61}$ | $\frac{30}{61}$ | $\frac{6}{61}$ |

2. Find the moment generating function of Poisson distribution. (A/M 2015) (N/D 2014)(A/M2017)

**Solution:**

$$\begin{aligned} M_X(t) &= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \lambda^x}{x!} = \sum_{x=0}^{\infty} (\lambda e^t)^x \frac{e^{-\lambda}}{x!} = e^{-\lambda} \left\{ 1 + \frac{\lambda e^t}{1!} + \frac{(\lambda e^t)^2}{2!} + \dots \right\} \\ &= e^{-\lambda} e^{\lambda e^t} = e^{-\lambda(e^t - 1)} \end{aligned}$$

3. Show that the function  $f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$  is a probability density function of a random variable X. (MA6451 A/M2015, N/D 2016)

**Solution:**

$$\int_{-\infty}^{\infty} f(x) dx = \int_0^{\infty} e^{-x} dx = \left[ -e^{-x} \right]_0^{\infty} = 0 - (-1) = 1$$

4. The mean and variance of binomial distribution are 5 and 4. Determine the distribution. (MA6451 A/M2015)

**Solution:**

mean =  $np = 5$  and variance =  $npq = 4$ .

$$\frac{npq}{np} = \frac{4}{5} \Rightarrow q = \frac{4}{5}$$

$$p = 1 - \frac{4}{5} = \frac{1}{5}, n(1/5) = 5, n = 25.$$

$$P(X = x) = nC_x p^x q^{n-x} = 25C_x \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{25-x}.$$

5. Given the probability density function  $f(x) = \frac{C}{1+x^2}$ ,  $-\infty < x < \infty$ , find k and C.D.F. (N/D 2014)

**Solution:**

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{C}{1+x^2} dx = 1$$

$$\Rightarrow C \left[ \tan^{-1} x \right]_{-\infty}^{\infty} = 1$$

$$\Rightarrow C \left[ \left[ \tan^{-1} \infty \right] - \left[ \tan^{-1} - \infty \right] \right] = 1$$

$$\Rightarrow C \left[ \frac{\pi}{2} + \frac{\pi}{2} \right] = 1$$

$$\Rightarrow C = \frac{1}{\pi}$$

$$F(x) = \int_{-\infty}^x f(x) dx$$

$$= \int_{-\infty}^x \frac{C}{1+x^2} dx$$

$$= \frac{1}{\pi} \left[ \tan^{-1} x \right]_{-\infty}^x$$

$$= \frac{1}{\pi} \left[ \left[ \tan^{-1} \infty \right] - \left[ \tan^{-1} - x \right] \right]$$

$$= \frac{1}{\pi} \left( \frac{\pi}{2} - \tan^{-1} x \right) = \frac{1}{\pi} \cot^{-1} x$$

6. Define random variable. (N/D 2013)

**Solution:**

A random variable is a function  $X: S \rightarrow R$  that assigns a real number  $X(S)$  to every element  $s \in S$ , where  $S$  is the sample space corresponding to a random experiment  $E$ .

**Ex:** Consider an experiment of tossing an unbiased coin twice. The outcomes of the experiment are HH, HT, TH, TT. Let  $X$  denote the number of heads turning up. Then  $X$  has the values 2, 1, 1, 0. Here  $X$  is a random variable which assigns a real number to every outcome of a random experiment.

7. Define geometric distribution. (N/D 2013)

**Solution:**

The geometric distribution is defined as,  $P(X = x) = q^{x-1} p$   $x = 1, 2, 3, \dots$

8.  $X$  and  $Y$  are independent random variables with variance 2 and 3. Find the variance of  $3X + 4Y$ . (M/J 2014)

**Solution:**

$$\begin{aligned}
V(3X + 4Y) &= 9\text{Var}(X) + 16\text{Var}(Y) + 24\text{Cov}(XY) \\
&= 9 \times 2 + 16 \times 3 + 0 \quad (\because X \text{ \& } Y \text{ are independent cov}(XY) = 0) \\
&= 18 + 48 = 66.
\end{aligned}$$

9. A Continuous random variable  $X$  has a probability density function  $f(x) = \begin{cases} 3x^2, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$ . Find 'a' such that  $P(X > a) = 0.5$ . (M/J 2014) (N/D 2010)(N/D 2016)

**Solution:**

$$P(X > a) = 0.5$$

$$\int_a^1 3x^2 dx = 0.5 \Rightarrow 3 \left( \frac{x^3}{3} \right)_a^1 = 0.5$$

$$1 - a^3 = 0.5 \Rightarrow$$

$$a^3 = \frac{1}{2}$$

$$a = \left( \frac{1}{2} \right)^{1/3}$$

10. A random variable  $X$  has cdf  $F(x) = \begin{cases} 0 & ; x < 1 \\ \frac{1}{2}(x^2 - 1) & ; 1 \leq x \leq 3 \\ 1 & ; x \geq 3 \end{cases}$ . Find the pdf of  $X$  and expected value of  $X$ . (M/J2013)(N/D 2017)

**Solution:**

$$f(x) = \frac{d}{dx} F(x) = x, \quad 1 \leq x \leq 3$$

$$E(X) = \int_0^{\infty} xf(x) dx = \int_1^3 xxdx$$

$$= \left[ \frac{x^3}{3} \right]_1^3$$

$$= \frac{1}{3}(27 - 1) = \frac{26}{3}$$

11. Find the moment generating function of binomial distribution. (M/J2013)(A/M2017)

**Solution:**

$$M_x(t) = \sum_{i=0}^n e^{ix} {}_n C_x p^x q^{n-x} = \sum_{x=0}^n {}_n C_x (pe^t)^x q^{n-x} = (q + pe^t)^n$$

12. If  $X$  is uniformly distributed in  $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ . Find the pdf of  $Y = \tan X$ . (N/D 2010)

**Solution:**

$$\text{Given } Y = \tan X \Rightarrow x = \tan^{-1} y$$

$$\therefore \frac{dx}{dy} = \frac{1}{1+y^2}$$

Since  $X$  is uniformly distribution in  $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ,

$$f_X(x) = \frac{1}{b-a} = \frac{1}{\frac{\pi}{2} + \frac{\pi}{2}}$$

$$f_X(x) = \frac{1}{\pi}, -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Now  $f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right| = \frac{1}{\pi} \left( \frac{1}{1+y^2} \right), -\infty < y < \infty$

$$\therefore f_Y(y) = \frac{1}{\pi(1+y^2)}, -\infty < y < \infty$$

13. The CDF of a continuous random variable is given by  $F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x/5}, & 0 \leq x < \infty \end{cases}$ .

Find the PDF and mean of  $X$ . (A/M 2011)

**Solution:**

$$f(x) = \frac{d}{dx} F(x) = \frac{1}{5} e^{-x/5}, x \geq 0$$

$$E(X) = \int_0^{\infty} x f(x) dx = \int_0^{\infty} \frac{1}{5} x e^{-x/5} dx$$

$$= \frac{1}{5} \left[ -x e^{-x/5} - e^{-x/5} \right]_0^{\infty}$$

$$= \frac{1}{5} (0 + 25) = 5$$

14. Establish the memoryless property of the exponential distribution. (A/M 2011)

**Solution:**

If  $X$  has a geometric distribution, then for any two positive integer  $s$  and  $t$ ,  
 $P(X > s + t | X > s) = P(X > t)$ .

15. A continuous random variable  $X$  has the probability density function  $f(x)$  given by  $f(x) = ce^{-|x|}$ ,  $-\infty < x < \infty$ . Find the value of  $c$  and cdf of  $X$ . (N/D 2005)

**Solution:**

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$\int_{-\infty}^{\infty} ce^{-|x|} dx = 1$$

$$2 \int_0^{\infty} ce^{-x} dx = 1$$

$$2 \int_0^{\infty} ce^{-x} dx = 1$$

$$2c(-e^{-x})_0^{\infty} = 1$$

$$2c(1) = 1$$

$$c = \frac{1}{2}$$

Case (i):  $x < 0$

$$\begin{aligned} F(x) &= \int_{-\infty}^x f(x) dx \\ &= \int_{-\infty}^x ce^{-|x|} dx \\ &= c \int_{-\infty}^x e^x dx \\ &= c(e^x)_{-\infty}^x \\ &= \frac{1}{2} e^x \end{aligned}$$

Case (ii):  $x > 0$

$$\begin{aligned} F(x) &= \int_{-\infty}^x f(x) dx \\ &= \int_{-\infty}^x ce^{-|x|} dx \\ &= c \int_{-\infty}^0 e^x dx + c \int_0^x e^{-x} dx \\ &= c(e^x)_0^0 + c(-e^{-x})_0^x \end{aligned}$$



$$\begin{aligned}
&= c + c(-e^{-x} + 1) \\
&= \frac{1}{2}(2 - e^{-x}) \\
F(x) &= \begin{cases} \frac{1}{2}e^x & , x > 0 \\ \frac{1}{2}(2 - e^{-x}) & , x < 0 \end{cases}
\end{aligned}$$

16. Is the function defined as follows a density function?(N/D 2006) (M/J 2012)

$$f(x) = \begin{cases} 0, & x < 2 \\ \frac{1}{18}(3 + 2x), & 2 \leq x \leq 4. \\ 0, & x > 4 \end{cases}$$

**Solution:**

$$\int_2^4 f(x) dx = \int_2^4 \frac{1}{18}(3 + 2x) dx = \left[ \frac{(3 + 2x)^2}{72} \right]_2^4 = 1$$

Hence it is density function.

17. The cumulative distribution function (CDF) of a random variable  $X$  is  $F(x) = 1 - (1 + x)e^{-x}$ ,  $x > 0$ . Find the probability density function of  $X$ . (M/J 2006)

**Solution:**

$$\begin{aligned}
f(x) &= F'(x) \\
&= 0 - [(1 + x)(-e^{-x}) + (1)(e^{-x})] \\
&= xe^{-x}, x > 0
\end{aligned}$$

18. The no. of monthly breakdowns of a computer is a r.v. having poisson distribution with mean 1.8. Find the probability that this computer will function for a month with only one breakdown. (M/J 2006)

**Solution:**

$$\begin{aligned}
p(X = x) &= \frac{e^{-\lambda} \lambda^x}{x!}, \text{ given } \lambda = 1.8 \\
p(x = 1) &= \frac{e^{-1.8} (1.8)^1}{1!} = 0.2975
\end{aligned}$$

19. Find the MGF of triangular distribution whose density function is given by

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & \text{elsewhere} \end{cases} . \text{ (M/J 2006)}$$

**Solution:**

$$\begin{aligned}
 M_X(t) &= E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f(x) dx \\
 &= \int_0^1 e^{tx} x dx + \int_1^2 e^{tx} (2-x) dx \\
 &= \left[ x \frac{e^{tx}}{t} - \frac{e^{tx}}{t^2} \right]_0^1 + \left[ (2-x) \frac{e^{tx}}{t} - (-1) \frac{e^{tx}}{t^2} \right]_1^2 \\
 &= \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2} \\
 M_X(t) &= \frac{e^{2t} - 2e^t + 1}{t^2}
 \end{aligned}$$

20. Show that for the uniform distribution  $f(x) = \frac{1}{2a}$ ,  $-a < x < a$ , the mgf about origin is

$$\frac{\sinh at}{at} \quad (\text{N/D 2006})$$

**Solution:**

$$\begin{aligned}
 \text{Given } f(x) &= \frac{1}{2a}, -a < x < a \\
 \text{MGF } M_x(t) &= E[e^{tx}] \\
 &= \int_{-\infty}^{\infty} e^{tx} f(x) dx = \int_{-a}^a e^{tx} \frac{1}{2a} dx \\
 &= \frac{1}{2a} \int_{-a}^a e^{tx} dx = \frac{1}{2a} \left[ \frac{e^{tx}}{t} \right]_{-a}^a \\
 &= \frac{1}{2at} [e^{at} - e^{-at}] = \frac{1}{2at} 2 \sinh at = \frac{\sinh at}{at} \\
 M_x(t) &= \frac{\sinh at}{at}
 \end{aligned}$$

21. If X has an exponential distribution with parameter  $\alpha$ , find the pdf of  $y = \log x$ . (N/D 2006)

**Solution:**

$$\text{The pdf of exponential distribution is } f(x) = \alpha e^{-\alpha x},$$

The pdf of Y is given by  $f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$  where  $y = \log x$

$$y = \log x \Rightarrow x = e^y, \quad \frac{dx}{dy} = e^y \Rightarrow \left| \frac{dx}{dy} \right| = e^y$$

$$\therefore f_Y(y) = \alpha e^{-\alpha x} e^y,$$

$$\Rightarrow f_Y(y) = \alpha e^{-\alpha e^y} e^y, \quad -\infty < y < \infty$$

22. Define exponential density function and find its mean and variance. (N/D 2005)

**Solution:**

The density function of exponential distribution is given by

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$$\begin{aligned} \text{Mean} = E[x] &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_0^{\infty} x \lambda e^{-\lambda x} dx = \lambda \int_0^{\infty} x e^{-\lambda x} dx \\ &= \lambda \left[ \frac{-x e^{-\lambda x}}{\lambda} - \frac{e^{-\lambda x}}{\lambda^2} \right]_0^{\infty} \\ &= \lambda \left[ (0-0) - \left( 0 - \frac{1}{\lambda^2} \right) \right] = \lambda \left( \frac{1}{\lambda^2} \right) = \frac{1}{\lambda} \end{aligned}$$

$$\text{Mean} = \frac{1}{\lambda}$$

$$\begin{aligned} E[x^2] &= \int_{-\infty}^{\infty} x^2 f(x) dx \\ &= \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx = \lambda \int_0^{\infty} x^2 e^{-\lambda x} dx \\ &= \lambda \left[ \frac{-x^2 e^{-\lambda x}}{\lambda} - \frac{2x e^{-\lambda x}}{\lambda^2} - \frac{2e^{-\lambda x}}{\lambda^3} \right]_0^{\infty} \\ &= \lambda \left[ (0-0-0) - \left( 0-0-\frac{2}{\lambda^3} \right) \right] = \lambda \left( \frac{2}{\lambda^3} \right) = \frac{2}{\lambda^2} \end{aligned}$$

$$\begin{aligned} \text{Variance} &= E(x^2) - [E(x)]^2 \\ &= \frac{2}{\lambda^2} - \left( \frac{1}{\lambda} \right)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} \end{aligned}$$

23. If the pdf of X is  $f_X(x) = 2x, 0 < x < 1$ , then find the pdf of  $Y = 3x + 1$ . (M/J 2007)

**Solution:**

The pdf of Y is given by  $f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$ , where  $Y = 3x + 1$

$$Y = 3x + 1 \Rightarrow x = \frac{y-1}{3}$$

$$\frac{dx}{dy} = \frac{1}{3} \Rightarrow \left| \frac{dx}{dy} \right| = \frac{1}{3}$$

$$\therefore f_Y(y) = 2x \cdot \frac{1}{3} = \frac{2}{3} \left( \frac{y-1}{3} \right) = \frac{2}{9}(y-1), 1 < y < 4$$

24. Define Discrete Random Variable. (A/M 2010)

**Solution:**

If X is a random variable which can take a finite number or countably infinite number of values, X is called a discrete RV.

Ex. Let X represent the sum of the numbers on the 2 dice, when two dice are thrown.

25. Write the M.G.F of Gamma distribution. (N/D 2004)(M/J 2007)

**Solution:**

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x) dx$$

$$= \frac{\lambda^\gamma}{\Gamma \gamma} \int_0^{\infty} e^{-(\lambda-t)x} x^{\gamma-1} dx = \frac{\lambda^\gamma}{\Gamma \gamma} \frac{\Gamma \gamma}{(\lambda-t)^\gamma}$$

$$\therefore M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-\gamma}$$

26. A discrete r.v X has mgf  $M_X(t) = e^{3(e^t-1)}$ . Find E(x), var(x), and P(x=0). (N/D 2012)

**Solution:**

$$\text{Given } M_X(t) = e^{3(e^t-1)}$$

We know that mgf of poisson is  $M_X(t) = e^{\lambda(e^t-1)}$

Therefore  $\lambda=3$

In poisson  $E(x) = \text{var}(x) = \lambda$

$\therefore \text{Mean } E(x) = \text{var}(x) = 3$

$$p(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$\therefore p(X = 0) = \frac{e^{-\lambda} \lambda^0}{0!} = e^{-\lambda} = e^{-3} = 0.123$$

27. An experiment, succeeds twice as often as it fails. Find the chance that in the next 4 trials, there shall be at least one success. (N/D 2012)

**Solution:**

$p=2/3, q=1/3, n=4$

$$P(X \geq 1) = 4C_1 \left(\frac{2}{3}\right)^1 \left(\frac{1}{3}\right)^3 + 4C_2 \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^2 + 4C_3 \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^1 + 4C_4 \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^0 = \frac{80}{81}$$

**UNIT I**  
**PART B**

1. A random variable X has cdf  $F(x) = \begin{cases} 0 & \text{if } x < -1 \\ \alpha(1+x) & \text{if } -1 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$ . Find the value of  $\alpha$ .  
Find  $P(X > 1/4)$  and  $P(-0.5 \leq X \leq 0)$ . (A/M 2015)
2. Obtain the moment generating function of geometric distribution. Hence, find its mean and variance. (A/M 2015)
3. If X is uniformly distributed with  $E(X)=1$  and  $\text{var}(X)=4/3$ , find  $P(X < 0)$ . (A/M 2015)
4. Obtain the moment generating function of exponential distribution. Hence compute the first four moments. (A/M 2015)(A/Y 2016)
5. A continuous random variable X that can assume any value between  $X=2$  and  $X=5$  has a pdf given by  $f(x)=k(1+x)$ . Find  $P(X < 4)$ . (MA6451 A/M2015)
6. If the probability that an applicant for a driver's license will pass the road tes on any given trial is 0.8, what is the probability that he will finally pass the test on the 4<sup>th</sup> trial? Also find the probability that he will finally pass the est in less than 4 trials. (MA6451 A/M2015)
7. Find the moment generating function of exponential distribution and hence find the mean and variance of exponential distribution. (MA6451 A/M2015)
8. If the probability mass function of a random variable X is given by  $P[X=x]=kx^3$ ,  $x=1,2,3,4$ , find the value of k,  $P[(1/2 < X < 5/2) | X > 1]$ , mean and variance of X. (MA6451 A/M2015)
9. Derive Poisson distribution from binomial distribution. (N/D 2013) (N/D 2014)
10. Find mean and variance of Gamma distribution. (N/D 2013) (N/D 2014)(N/D 2016)
11. If X and Y are independent RVs each normally distributed with mean zero and variance  $\sigma^2$ , find the pdf of  $R = \sqrt{X^2 + Y^2}$  and  $\varphi = \tan^{-1} \left( \frac{Y}{X} \right)$ . (N/D 2013)
12. Find the n<sup>th</sup> moment about mean of normal distribution. (N/D 2014)

13. If a random variable has the probability density  $f(x) = \begin{cases} 2e^{-2x} & , x \geq 0 \\ 0 & , \text{Otherwise} \end{cases}$ . Obtain mgf and first four moments about origin. Find mean and variance of the same. (N/D 2014)

14. A random variable X has the following probability distribution.

|       |   |   |    |    |    |                |                 |                    |
|-------|---|---|----|----|----|----------------|-----------------|--------------------|
| X:    | 0 | 1 | 2  | 3  | 4  | 5              | 6               | 7                  |
| f(x): | 0 | k | 2k | 2k | 3k | k <sup>2</sup> | 2k <sup>2</sup> | 7k <sup>2</sup> +k |

Find (i) the value of k (ii)  $P(1.5 < X < 4.5 \mid X > 2)$  (iii) Find the smallest value of n for which  $P(X \leq n) > \frac{1}{2}$ . (M/J 2014) (N/D 2010)

15. Find the mgf of the random variable X having the pdf  $f(x) = \begin{cases} \frac{x}{4} e^{-x/2}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$ . Also deduce the first four moments about origin. (M/J 2014)

16. Define moment generating function of a random variable. Derive MGF, mean variance and the first four moments of Gamma distribution. (M/J 2014) (M/J2013)(N/D 2016)

17. Define Binomial distribution. Obtain the moment generating function. Hence compute first four moments and the recursion relation for the central moments. (M/J 2014)

18. A random variable X has pdf  $f(x) = \begin{cases} kx^2 e^{-x}; & x > 0 \\ 0 & \text{otherwise} \end{cases}$ . Find the r<sup>th</sup> moment of X about origin. Hence find the mean and variance. (M/J2013)

19. A random variable X is uniformly distributed over (0,10). Find (i)  $P(X < 3), P(X > 7)$  and  $P(2 < X < 5)$  (ii)  $P(X=7)$ . (M/J2013)

20. Find the M.G.F. of the random variable X having the probability density function  $f(x) = \begin{cases} \frac{x}{4} e^{-x/2}, & x > 0 \\ 0, & \text{else where} \end{cases}$ . Also deduce that first four moments about the origin. (N/D 2010)

21. An office has four phone lines. Each is busy about 10% of the time. Assume that the phone lines act independently.

- (i) What is the probability that all four phones are busy?
- (ii) What is the probability that atleast two of them are busy? (M/J2013)

22. If X is uniformly distributed in (-1,1) then find the pdf of  $Y = \sin \frac{\pi X}{2}$ . (N/D 2010)

23. If X and Y are independent random variables following  $N(8, 2)$  and  $N(12, 4\sqrt{3})$  respectively. Find the value of  $\lambda$  such that  $P[2X - Y \leq 2\lambda] = P[X + 2Y \geq \lambda]$ . (N/D 2010)

**UNIT II**  
**TWO - DIMENSIONAL RANDOM VARIABLES**

**PART A**

1. The joint pdf of (X,Y) is given by  $f(x, y) = kxye^{-(x^2+y^2)}$ ;  $x>0, y>0$ . Find the value of k. (A/M 2015) (MA6451 A/M2015) (N/D 2013)

**Solution:**

Given  $f(x, y)$  is the joint pdf, we have

$$\iint f(x, y) dx dy = 1 \quad \text{put } x^2 = t$$

$$\int_0^{\infty} \int_0^{\infty} kxye^{-(x^2+y^2)} dx dy = 1 \quad 2x dx = dt$$

$$k \int_0^{\infty} \int_0^{\infty} xye^{-x^2} e^{-y^2} dx dy = 1 \quad x dx = \frac{dt}{2}$$

$$k \int_0^{\infty} ye^{-y^2} \left[ \int_0^{\infty} xe^{-x^2} dx \right] dy = 1 \quad \text{when } x=0, t=0 \text{ and when } x=\infty, t=\infty$$

$$k \int_0^{\infty} ye^{-y^2} \left[ \int_0^{\infty} e^{-t} \frac{dt}{2} \right] dy = 1$$

$$\frac{k}{2} \int_0^{\infty} ye^{-y^2} (-e^{-t})_0^{\infty} dy = 1$$

$$\frac{k}{2} \int_0^{\infty} ye^{-y^2} (0+1) dy = 1 \quad \text{put } y^2 = t$$

$$\frac{k}{2} \int_0^{\infty} e^{-t} \frac{dt}{2} = 1 \quad 2y dy = dt$$

$$\frac{k}{4} (e^{-t})_0^{\infty} = 1 \quad y dy = \frac{dt}{2}$$

$$\frac{k}{4} (0+1) = 1 \quad \text{when } y=0, t=0 \text{ and when } y=\infty, t=\infty$$

$$\frac{k}{4} = 1$$

Therefore, the value of  $k$  is  $k=4$ .

2. Define the distribution function of two dimensional random variable (X,Y). State any one property. (A/M 2015)

**Solution:**

If (X,Y) is a two dimensional RV the  $F(x, y) = P(X \leq x, Y \leq y)$  is called joint CDF of (X,Y).

Properties of joint distribution of  $(X, Y)$  are

- (i)  $F[-\infty, y]=0=F[x, -\infty]$  and  $F[-\infty, \infty]=1$
- (ii)  $P[a < X < b, Y \leq y] = F(b, y) - F(a, y)$
- (iii)  $P[X \leq x, c < Y < d] = F[x, d] - F[x, c]$
- (iv)  $P[a < X < b, c < Y < d] = F[b, d] - F[a, d] - F[b, c] + F[a, c]$
- (v) At point  $s$  of continuity of  $f(x, y)$ ,  $\frac{\partial^2 F}{\partial x \partial y} = f(x, y)$

3. What is the angle between the two regression lines? (MA6451 A/M2015) (MA6451 A/M2015) (A/M 2011) (N/D 2012)(A/M 2016)

**Solution:**

The slopes of the regression lines are

$$m_1 = r \frac{\sigma_y}{\sigma_x}, m_2 = \frac{1}{r} \frac{\sigma_y}{\sigma_x}$$

If  $\theta$  is the angle between the lines, Then

$$\tan \theta = \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2} \left[ \frac{1 - r^2}{r} \right]$$

When  $r=0$ , that is when there is no correlation between  $x$  and  $y$ ,  $\tan \theta = \infty$  (or)  $\theta = \frac{\pi}{2}$  and so the regression lines are perpendicular. When  $r=1$  or  $r=-1$ , that is when there is a perfect correlation +ve or -ve,  $\theta = 0$  and so the lines coincide.

4. Given the RV  $X$  with density function  $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$ . Find the pdf of  $y=8x^3$ . (N/D 2013) (N/D 2014)

**Solution:**

The pdf of  $y$  is given by

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

Where  $y = 8x^3$

$$x^3 = \frac{y}{8} \Rightarrow x = \left( \frac{y}{8} \right)^{\frac{1}{3}}$$

$$\frac{dx}{dy} = \frac{1}{3} \left( \frac{y}{8} \right)^{\frac{1}{3}-1} \frac{1}{8} = \frac{1}{24} \left( \frac{y}{8} \right)^{-\frac{2}{3}}$$



$$\begin{aligned}
f_Y(y) &= 2x \frac{1}{24} \left(\frac{y}{8}\right)^{-\frac{2}{3}} \\
&= \frac{2}{24} \left(\frac{y}{8}\right)^{\frac{1}{3}} \left(\frac{y}{8}\right)^{-\frac{2}{3}} = \frac{2}{24} \left(\frac{y}{8}\right)^{-\frac{1}{3}} \\
&= \frac{1}{12} \left(\frac{8}{y}\right)^{\frac{1}{3}} = \frac{1}{6} (y)^{-\frac{1}{3}} \quad , \\
0 < x < 1 &\Rightarrow 0 < y < 8 \\
f_Y(y) &= \frac{1}{6} (y)^{-\frac{1}{3}} \quad , 0 < y < 8
\end{aligned}$$

5. Define joint pmf of a two dimensional discrete random variable. (N/D 2014)

**Solution:**

Let  $(X, Y)$  be a two dimensional discrete random variable. Let  $P(X=x_i, Y=y_j)=p_{ij}$ .  $p_{ij}$  is called the probability function of  $(X, Y)$  or joint probability distribution. If the following conditions are satisfied

1.  $p_{ij} \geq 0$  for all  $i$  and  $j$
2.  $\sum_j \sum_i p_{ij} = 1$

The set of triples  $(x_i, y_j, p_{ij})$   $i=1, 2, 3, \dots$  and  $j=1, 2, 3, \dots$  is called the Joint probability distribution of  $(X, Y)$ .

6. State central limit theorem for iid random variables. (M/J 2014) (N/D 2010)

**Solution:**

If  $X_1, X_2, X_3, \dots, X_n, \dots$  be a sequence of independent identically distributed random variables with  $E[X_i]=\mu$  and  $Var(X_i)=\sigma^2, i=1, 2, \dots$ , and if  $S_n = X_1 + X_2 + X_3 + \dots + X_n$ , then under certain general conditions,  $S_n$  follows a normal distribution with mean  $n\mu$  and variance  $n\sigma^2$  as  $n$  tends to infinity.

7. State the basic properties of joint distribution of  $(X, Y)$  where  $X$  and  $Y$  are random variables. (M/J 2014)

**Solution:**

Properties of joint distribution of  $(X, Y)$  are

- (i)  $F[-\infty, y]=0=F[x, -\infty]$  and  $F[-\infty, \infty]=1$
- (ii)  $P[a < X < b, Y \leq y] = F(b, y) - F(a, y)$
- (iii)  $P[X \leq x, c < Y < d] = F[x, d] - F[x, c]$
- (iv)  $P[a < X < b, c < Y < d] = F[b, d] - F[a, d] - F[b, c] + F[a, c]$
- (v) At point  $s$  of continuity of  $f(x, y)$ ,  $\frac{\partial^2 F}{\partial x \partial y} = f(x, y)$

8. The joint pdf of a two dimensional random variable  $(X, Y)$  is given by

$$f(x, y) = xy^2 + \frac{x^2}{8}, 0 \leq x \leq 2; 0 \leq y \leq 1. \text{ Compute } P[X < Y]. \text{ (M/J2013)(A/M 2009)}$$

**Solution:**

$$P[X < Y] = \int_0^1 \int_0^y \left( xy^2 + \frac{x^2}{8} \right) dx dy = \int_0^1 \left[ y^2 \left( \frac{x^2}{2} \right)_0^y + \frac{1}{8} \left( \frac{x^3}{3} \right)_0^y \right] dy$$

$$= \int_0^1 \left[ \frac{y^2}{2} (y^2 - 0) + \frac{1}{24} (y^3 - 0) \right] dy = \int_0^1 \left[ \frac{y^4}{2} + \frac{y^3}{24} \right] dy$$

$$= \frac{1}{2} \left( \frac{y^5}{5} \right)_0^1 + \frac{1}{24} \left( \frac{y^4}{4} \right)_0^1 = \frac{1}{10} (1-0) + \frac{1}{96} (1-0) = \frac{1}{10} + \frac{1}{96} = \frac{53}{480}$$

9. The joint pmf of two random variables X and Y is given by

$$p(x, y) = \begin{cases} kxy, & x = 1, 2, 3; y = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}. \text{ Determine the value of the constant } k.$$

(M/J2013)(A/M 2016)

**Solution:**

The joint probability distribution of  $(X, Y)$  is given below

| X \ Y | 1  | 2  | 3  |
|-------|----|----|----|
| 1     | 1k | 2k | 3k |
| 2     | 2k | 4k | 6k |
| 3     | 3k | 6k | 9k |

Since  $p(x, y)$  is a probability mass function, we have

$$\begin{aligned} \sum \sum p(x, y) &= 1 \\ k + 2k + 3k + 2k + 4k + 6k + 3k + 6k + 9k &= 1 \\ 36k &= 1 \\ k &= \frac{1}{36} \end{aligned}$$

10. Let  $X$  and  $Y$  be continuous random variables with joint probability density function

$$f_{XY}(x, y) = \begin{cases} \frac{x(x-y)}{8}, & 0 < x < 2, -x < y < x \\ 0 & \text{otherwise} \end{cases}. \text{ Find } f_{Y/X}(y/x). \text{ (N/D 2010) (A/M 2011)}$$

**Solution:**

$$f(x) = \frac{1}{8} \int_{-x}^x x(x-y) dy = \frac{x}{8} \left( xy - \frac{y^2}{2} \right)_{-x}^x = \frac{x^3}{4}, 0 < x < 2$$

$$f_{Y/X}(y/x) = \frac{f(x,y)}{f(x)} = \frac{1}{2x^2} (x-y), -x < y < x.$$

11. Given the joint pdf of  $(X, Y)$   $f(x, y) = \begin{cases} e^{-(x+y)}, & x > 0, y > 0 \\ 0 & \text{elsewhere} \end{cases}$ . Find the marginal densities of  $X$  and  $Y$ . Are  $X$  and  $Y$  independent? (A/M 2008)(N/D 2016)

**Solution:**

Marginal density of  $X$  is

$$\begin{aligned} f_X(x) &= \int f(x, y) dy \\ &= \int_0^{\infty} e^{-x} e^{-y} dy = e^{-x} \int_0^{\infty} e^{-y} dy = e^{-x} (-e^{-y})_0^{\infty} \\ &= -e^{-x} (0-1) = e^{-x}, x > 0 \end{aligned}$$

Marginal density of  $Y$  is

$$\begin{aligned} f_Y(y) &= \int f(x, y) dx \\ &= \int_0^{\infty} e^{-x} e^{-y} dx = e^{-y} \int_0^{\infty} e^{-x} dx = e^{-y} (-e^{-x})_0^{\infty} \\ &= -e^{-y} (0-1) = e^{-y}, y > 0 \end{aligned}$$

$$f_X(x) \cdot f_Y(y) = e^{-x} \cdot e^{-y} = e^{-(x+y)} = f_{XY}(x, y)$$

Therefore  $X$  and  $Y$  are independent.

12. A r.v.  $X$  has pdf  $f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0 & , x < 0 \end{cases}$  find the density function of  $\frac{1}{x}$  (M/J 2009)

**Solution:**

The pdf of  $Y$  is given by  $f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$  where  $y = \frac{1}{x}$

$$y = \frac{1}{x} \Rightarrow x = \frac{1}{y} \qquad \frac{dx}{dy} = -\frac{1}{y^2} \Rightarrow \left| \frac{dx}{dy} \right| = \frac{1}{y^2}$$

$$f_Y(y) = e^{-x} \frac{1}{y^2} = \frac{e^{-1/y}}{y^2}, y > 0$$

13. If the function  $f(x, y) = c(1-x)(1-y)$ ,  $0 < x < 1$ ,  $0 < y < 1$  is to be a density function, find the value of  $c$ . (A/M 2008)

**Solution:**

Given  $f(x, y)$  is the joint pdf, we have

$$\begin{aligned} \iint f(x, y) dx dy &= 1 \\ \int_0^1 \int_0^1 c(1-x)(1-y) dx dy &= 1 \\ c \int_0^1 \int_0^1 (1-x-y+xy) dx dy &= 1 \\ c \int_0^1 \left[ (x)_0^1 - \left( \frac{x^2}{2} \right)_0^1 - y(x)_0^1 + y \left( \frac{x^2}{2} \right)_0^1 \right] dy &= 1 \\ c \int_0^1 \left[ 1 - \frac{1}{2} - y + \frac{y}{2} \right] dy &= 1 \\ c \int_0^1 \left( \frac{1}{2} - \frac{y}{2} \right) dy &= 1 \\ c \left[ \frac{1}{2}(y)_0^1 - \frac{1}{2} \left( \frac{y^2}{2} \right)_0^1 \right] &= 1 \Rightarrow c \left[ \frac{1}{2} - \frac{1}{4} \right] = 1 \Rightarrow \frac{c}{4} = 1 \Rightarrow c = 4 \end{aligned}$$

Therefore the value of  $c$  is  $c = 4$

14. Find the marginal density functions of  $X$  and  $Y$  if  $f(x, y) = \frac{2}{5}(2x+5y)$ ,  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ .

(N/D 2006)

**Solution:**

Marginal density of  $X$  is

$$\begin{aligned} f_x(x) &= \int f(x, y) dy \\ &= \frac{2}{5} \int_0^1 (2x+5y) dy = \frac{2}{5} \left[ 2x(y)_0^1 + 5 \left( \frac{y^2}{2} \right)_0^1 \right] \\ &= \frac{2}{5} \left[ 2x + \frac{5}{2} \right] = \frac{4}{5}x + 1, \quad 0 \leq x \leq 1 \end{aligned}$$

Marginal density of  $Y$  is

$$\begin{aligned} f_y(y) &= \int f(x, y) dx \\ &= \frac{2}{5} \int_0^1 (2x+5y) dx = \frac{2}{5} \left[ 2 \left( \frac{x^2}{2} \right)_0^1 + 5y(x)_0^1 \right] \end{aligned}$$

$$= \frac{2}{5}[1+5y] = \frac{2}{5} + 2y, \quad 0 \leq y \leq 1$$

15. The joint pdf of the random variable  $(X, Y)$  is  $f(x, y) = \begin{cases} cxy & , 0 < x < 2 ; 0 < y < 2 \\ 0 & , otherwise \end{cases}$ .

Find the value of  $c$ . (N/D 2009)

**Solution:**

Given  $f(x, y)$  is the joint pdf, we have

$$\begin{aligned} \iint f(x, y) dx dy &= 1 \\ \int_0^2 \int_0^2 cxy dx dy &= 1 \\ c \int_0^2 \int_0^2 xy dx dy &= 1 \\ c \int_0^2 y \left( \frac{x^2}{2} \right)_0^2 dy &= 1 \\ c \int_0^2 y(2-0) dy &= 1 \\ 2c \left[ \frac{y^2}{2} \right]_0^2 &= 1 \\ c[4-0] = 1 &\Rightarrow 4c = 1 \Rightarrow c = \frac{1}{4} \end{aligned}$$

Therefore the value of  $c$  is  $c = \frac{1}{4}$ .

16. If the joint pdf of  $(X, Y)$  is given by  $f(x, y) = 2 - x - y$ ;  $0 \leq x < y \leq 1$ , find  $E[X]$ .

**Solution:**

$$\begin{aligned} E[X] &= \iint xf(x, y) dx dy \\ &= \int_0^1 \int_0^y x[2-x-y] dx dy \\ &= \int_0^1 \int_0^y (2x - x^2 - xy) dx dy = \int_0^1 \left[ 2 \left( \frac{x^2}{2} \right)_0^y - \left( \frac{x^3}{3} \right)_0^y - y \left( \frac{x^2}{2} \right)_0^y \right] dy \\ &= \int_0^1 \left( y^2 - \frac{y^3}{3} - \frac{y^3}{2} \right) dy = \int_0^1 \left( y^2 - \frac{5}{6} y^3 \right) dy = \left( \frac{y^3}{3} \right)_0^1 - \frac{5}{6} \left( \frac{y^4}{4} \right)_0^1 \end{aligned}$$

$$= \frac{1}{3} - \frac{5}{24} = \frac{3}{24} = \frac{1}{8}.$$

17. If  $Y = -2X + 3$ , find  $Cov(X, Y)$ . (A/M 2008)

**Solution:**

$$\begin{aligned} Cov(X, Y) &= E[XY] - E[X]E[Y] \\ &= E[X(-2X + 3)] - E[X]E[-2X + 3] \\ &= E[-2X^2 + 3X] - E[X](-2E[X] + 3) \\ &= -2E[X^2] + 3E[X] + 2(E[X])^2 - 3E[X] \\ &= -2[E[X^2] - (E[X])^2] = -2Var X. \end{aligned}$$

18. The joint probability mass function of  $(X, Y)$  is given by  $p(x, y) = k(2x + 3y)$ ,  $x = 0, 1, 2$ ,  $y = 1, 2, 3$ . Find the value of  $k$  and marginal distribution of  $X$ .

**Solution:**

The joint probability distribution of  $(X, Y)$  is given below

|       |    |     |     |
|-------|----|-----|-----|
| X \ Y | 1  | 2   | 3   |
| 0     | 3k | 6k  | 9k  |
| 1     | 5k | 8k  | 11k |
| 2     | 7k | 10k | 13k |

Since  $p(x, y)$  is a probability mass function, we have

$$\begin{aligned} \sum \sum p(x, y) &= 1 \\ 3k + 6k + 9k + 5k + 8k + 11k + 7k + 10k + 13k &= 1 \\ 72k &= 1 \\ k &= \frac{1}{72} \end{aligned}$$

**Marginal distribution of X:**

|             |       |       |       |
|-------------|-------|-------|-------|
| <b>X</b>    | 0     | 1     | 2     |
| <b>P(X)</b> | 18/72 | 24/72 | 30/72 |

19.  $X$  and  $Y$  are independent random variables with variance 2 and 3. Find the variance of  $3X + 4Y$ .

**Solution:**

$$\begin{aligned} V(3X + 4Y) &= 9Var(X) + 16Var(Y) + 24Cov(XY) \\ &= 9 \times 2 + 16 \times 3 + 0 \quad (\because X \text{ \& } Y \text{ are independent } cov(XY) = 0) \\ &= 18 + 48 = 66. \end{aligned}$$

20. The joint pdf of (X,Y) is  $f(x,y) = \frac{1}{4}$ ,  $0 \leq x,y < 2$ , Find  $P(X + Y \leq 1)$ .(N/D 2005)

**Solution:**

$$P[X + Y \leq 1] = \int_0^1 \int_0^{1-y} \frac{1}{4} dx dy = \frac{1}{4} \int_0^1 (1-y) dy = \frac{1}{8}.$$

21. Let X and Y be two discrete random variable with joint pmf

$$P[X = x, Y = y] = \begin{cases} \frac{x+2y}{18}, & x=1,2; y=1,2 \\ 0 & , otherwise \end{cases}. \text{ Find the marginal pmf of } X \text{ and } E[X]. \text{ (M/J}$$

2012)

**Solution:**

The joint pmf of (X, Y) is given by

|       |                |                |
|-------|----------------|----------------|
| X \ Y | 1              | 2              |
| 1     | $\frac{3}{18}$ | $\frac{4}{18}$ |
| 2     | $\frac{5}{18}$ | $\frac{6}{18}$ |

Marginal pmf of X is

$$P[X = 1] = \frac{3}{18} + \frac{5}{18} = \frac{8}{18} = \frac{4}{9}$$

$$P[X = 2] = \frac{4}{18} + \frac{6}{18} = \frac{10}{18} = \frac{5}{9}$$

$$E[X] = \sum x p(x) = (1)\left(\frac{4}{9}\right) + (2)\left(\frac{5}{9}\right) = \frac{4}{9} + \frac{10}{9} = \frac{14}{9}.$$

22. If X and Y have joint pdf  $f(x,y) = \begin{cases} x+y; & 0 < x < 1, 0 < y < 1 \\ 0 & ; otherwise \end{cases}$ . Check whether X and Y are independent. (N/D 2005)

$$f_x(x) = \int f(x,y) dy$$

$$= \int_0^1 (x+y) dy = x(y)_0^1 + \left(\frac{y^2}{2}\right)_0^1 = x + \frac{1}{2}, 0 < x < 1$$

$$f_y(y) = \int f(x,y) dx$$

$$= \int_0^1 (x+y) dx = \left(\frac{x^2}{2}\right)_0^1 + y(x)_0^1 = y + \frac{1}{2}, 0 < y < 1$$

$$f_X(x) \cdot f_Y(y) = \left(x + \frac{1}{2}\right) \left(y + \frac{1}{2}\right) = xy + \frac{x}{2} + \frac{y}{2} + \frac{1}{4} \neq x + y \neq f(x, y)$$

Therefore,  $X$  and  $Y$  are not independent variables.

23. Let  $X$  and  $Y$  be random variable with joint density function

$$f_{XY}(x, y) = \begin{cases} 4xy & ; 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & , \text{otherwise} \end{cases}. \text{ Find } E[XY]. (\text{M/J 2004})$$

**Solution:**

$$\begin{aligned} E[XY] &= \iint xy f(x, y) dx dy \\ &= \int_0^1 \int_0^1 xy (4xy) dx dy = 4 \int_0^1 \int_0^1 x^2 y^2 dx dy = 4 \int_0^1 y^2 \left(\frac{x^3}{3}\right)_0^1 dy \\ &= \frac{4}{3} \int_0^1 y^2 dy = \frac{4}{3} \left(\frac{y^3}{3}\right)_0^1 = \frac{4}{3} \left(\frac{1}{3}\right) = \frac{4}{9}. \end{aligned}$$

24. Write any two properties of regression coefficients. (M/J 2007)(N/D 2016)

**Solution:**

1. Correlation coefficient is the geometric mean of regression coefficients
2. If one of the regression coefficients is greater than unity then the other should be less than 1.

$$b_{xy} = r \frac{\sigma_y}{\sigma_x} \text{ and } b_{yx} = r \frac{\sigma_x}{\sigma_y}$$

If  $b_{xy} > 1$  then  $b_{yx} < 1$ .

25. The conditional pdf of  $X$  and  $Y$  is given by  $f\left(\frac{x}{y}\right) = \frac{x+y}{1+y} e^{-x}$ . Find  $P(X < 1 | Y = 2)$ . (N/D 2010)

**Solution:**

$$\text{When } y=2, f(x/y=2) = \frac{x+2}{3} e^{-x}$$

$$\therefore P[X < 1 | Y = 2] = \int_0^1 \frac{x+2}{3} e^{-x} dx = \frac{1}{3} \int_0^1 x e^{-x} dx + \frac{2}{3} \int_0^1 e^{-x} dx = 1 - \frac{4}{3} e^{-1}$$

24. Suppose the pdf  $f(x, y)$  of  $(X, Y)$  is given by  $f(x, y) = \begin{cases} \frac{6}{5}(x+y^2); & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & ; \text{otherwise} \end{cases}$ .

Obtain the marginal pdf of  $X$ . (N/D 2012)

**Solution:**

Marginal density of  $X$  is

$$\begin{aligned} f_X(x) &= \int f(x, y) dy = \int_0^1 \frac{6}{5}(x+y^2) dy \\ &= \frac{6}{5} \left[ x(y)_0^1 + \left(\frac{y^3}{3}\right)_0^1 \right] = \frac{6}{5} \left[ x + \frac{1}{3} \right], 0 \leq x \leq 1 \end{aligned}$$



**UNIT II**  
**PART B**

- Three balls are drawn at random without replacement from a box containing 2 white, 3 red and 4 blue balls. If  $X$  denotes the number of white balls drawn and  $Y$  denotes the number of red balls drawn, find the probability distribution of  $X$  and  $Y$ . (A/M 2015)
- The random variables  $X$  and  $Y$  are related by  $X-6=Y$  and  $0.64X-4.08=0$ . Find the mean of  $X$  and  $Y$  and correlation coefficient between  $X$  and  $Y$ . (A/M 2015)

- If the joint pdf of two dimensional random variable  $(X, Y)$  is given by

$$f(x, y) = \begin{cases} x^2 + \frac{xy}{3}, & 0 < x < 1, 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}. \text{ Find the marginal density function of } X \text{ and } Y.$$

Find the conditional density function of  $X$  given  $Y$ . (A/M 2015)

- A random sample of size 100 is taken from a population whose mean  $\mu = 60$  and variance  $\sigma^2 = 400$ . Using central limit theorem with what probability can we assert that the mean of the sample will not differ from  $\mu$  by more than 4. (A/M 2015)

- If the joint distribution functions of  $X$  and  $Y$  is given by

$$F(x, y) = \begin{cases} (1 - e^{-x})(1 - e^{-y}), & x > 0, y > 0 \\ 0 & , \text{ otherwise} \end{cases}$$

- Find the marginal density of  $X$  and  $Y$ .
  - Are  $X$  and  $Y$  independent.
  - $P(1 < X < 3, 1 < Y < 2)$ . (MA6451 A/M2015)(A/M 2016)
- Calculate the correlation coefficient for the following heights (in inches) of fathers  $X$  and their sons  $Y$ . (MA6451 A/M2015)

|     |    |    |    |    |    |    |    |    |
|-----|----|----|----|----|----|----|----|----|
| $X$ | 65 | 66 | 67 | 67 | 68 | 69 | 70 | 72 |
| $Y$ | 67 | 68 | 65 | 68 | 72 | 72 | 69 | 71 |

- The two lines of regression are  $8X-10y+66=0$ ,  $4X-18Y-214=0$ . The variance of  $X$  is 9. Find the mean value of  $X$  and  $Y$ . Also find the coefficient of correlation between the variables  $X$  and  $Y$ . (MA6451 A/M2015)

- Two random variables  $X$  and  $Y$  have the following joint pdf  
 $f(x, y) = \begin{cases} x + y, & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$ . Find the pdf of the random variable  $U=XY$ .  
 (MA6451 A/M2015)

- State and prove central limit theorem for iid RVs. (N/D2013) (M/J2013) (N/D 2014)

- If  $X$  and  $Y$  are independent RVs with pdfs  $e^{-x}; x \geq 0$  and  $e^{-y}; y \geq 0$

respectively. Find the pdf of  $U=X/X+Y$  and  $V=X+Y$ . Are U and V independent? (N/D2013)

11. The joint probability mass function of  $(X, Y)$  is given by  $p(x, y) = k(2x + 3y)$ ,  $x=0,1,2$   $y=1,2,3$ . Find all the marginal and conditional probability distributions. Also find the probability distribution of  $X + Y$ . (N/D2013) (N/D 2014)

12. The lifetime of a certain brand of an electric bulb may be considered a RV with mean  $1200h$  and standard deviation  $250h$ . Find the probability using central limit theorem, that the average lifetime of 60 bulbs exceeds  $1250h$ . (N/D 2014)

13. If the joint pdf of two dimensional random variable  $(X, Y)$  is given by

$f(x, y) = \begin{cases} x^2 + \frac{xy}{3}, & 0 < x < 1, 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$ . Find  $P(X > 1/2)$ ,  $P(Y < X)$ ,  $P(X + Y \geq 1)$ . Find the conditional density function. (M/J 2014)

14. The joint pdf of the random variable  $(X, Y)$  is  $f(x, y) = 3(x + y)$ ,  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ ,  $x + y \leq 1$ , Find  $COV(X, Y)$  (M/J 2014)

15. obtained by 10 students in Mathematics (x) and Statistics is given below

|   |    |    |    |    |    |    |    |    |    |    |
|---|----|----|----|----|----|----|----|----|----|----|
| X | 60 | 34 | 40 | 50 | 45 | 40 | 22 | 43 | 42 | 64 |
| Y | 75 | 32 | 33 | 40 | 45 | 33 | 12 | 30 | 34 | 51 |

Find the regression lines. Also find y when  $x=55$ . (M/J 2014)

16. Two independent random variables X and Y are defined by  $f_x(x) = \begin{cases} 4ax; & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$  and  $f_y(y) = \begin{cases} 4by; & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$ . Show that  $U=X+Y$  and  $V=X-Y$  are uncorrelated. (M/J2013)

17. The equations of two regression lines are  $3x+12y=19$  and  $3y+9x=46$ . Find mean of X and Y and the correlation coefficient between X and Y. (M/J2013)

18. Given the joint pdf of X and y  $f(x, y) = \begin{cases} Cx(x - y); & 0 < x < 2, -x < y < x \\ 0 & \text{otherwise} \end{cases}$ . (i) Evaluate C. (ii) Find marginal pdf of X. (iii) Find the conditional density of Y/X. (M/J2013)

19. Two random variables X and Y have the joint probability density function given by  $f_{XY}(x, y) = \begin{cases} k(1 - x^2y), & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$  (i) Find the value of k (ii) Obtain the marginal density function of X and Y. (iii) Also find the correlation coefficient between X and Y. (N/D 2010)

20. If X and Y are independent continuous random variables, show that the pdf of  $U = X + Y$  is given by  $h(u) = \int_{-\infty}^{\infty} f_x(v)f_y(u - v)dv$ . (N/D 2010)

21. If  $V_i$ ,  $i=1, 2, 3, \dots, 20$  are independent noise voltages received in an adder and  $V$  is the sum of the voltages received, find the probability that the total incoming voltage  $V$  exceeds 105, using the central limit theorem. Assume that each of the random variables  $V_i$  is uniformly distributed over  $(0,10)$ . (N/D 2010)

**UNIT III**  
**RANDOM PROCESSES**

**PART A**

1. Define a Markov process. (A/M 2015) (N/D 2013) (N/D 2014)

**Solution:**

Markov process is one in which the future value is independent of the past values, given the present value.

2. Prove that the sum of two independent Poisson processes is a Poisson process. (A/M 2015) (N/D 2012)

**Solution:**

Let  $X(t) = X_1(t) + X_2(t)$

$$\begin{aligned}
 P[X(t) = n] &= P[X_1(t) + X_2(t) = n] \\
 &= \sum_{r=0}^n P[X_1(t) = r]P[X_2(t) = n - r] \\
 &= \sum_{r=0}^n \frac{e^{-\lambda_1 t} (\lambda_1 t)^r}{r!} \frac{e^{-\lambda_2 t} (\lambda_2 t)^{n-r}}{(n-r)!} \\
 &= e^{-\lambda_1 t} e^{-\lambda_2 t} \sum_{r=0}^n \frac{\lambda_1^r t^r \lambda_2^{n-r} t^{n-r}}{r!(n-r)!} \\
 &= e^{-(\lambda_1 + \lambda_2)t} \sum_{r=0}^n \frac{n C_r}{n!} t^n \lambda_1^r \lambda_2^{n-r} \\
 &= e^{-(\lambda_1 + \lambda_2)t} \frac{t^n}{n!} [\lambda_2^n + n C_1 \lambda_2^{n-1} \lambda_1 + n C_2 \lambda_2^{n-2} \lambda_1^2 + \dots \dots \lambda_1^n] \\
 &= e^{-(\lambda_1 + \lambda_2)t} \frac{t^n}{n!} (\lambda_1 + \lambda_2)^n \\
 &= e^{-(\lambda_1 + \lambda_2)t} \frac{t^n}{n!} (\lambda_1 + \lambda_2)^n \\
 &= e^{-(\lambda_1 + \lambda_2)t} \frac{((\lambda_1 + \lambda_2)t)^n}{n!}
 \end{aligned}$$

Therefore,  $X_1(t) + X_2(t)$  is a Poisson process with parameter  $(\lambda_1 + \lambda_2)t$ .

3. Give an example of evolutionary random process. (MA6451 A/M2015)

**Solution:**

A Random processes that is not stationary in any sense is called an Evolutionary process.

Example: Poisson process.

4. Define a semi-random telegraph signal process. (MA6451 A/M2015)(N/D 2016)

**Solution:**

If  $N(t)$  represent the number of occurrences of a specified event in  $(0,t)$  and  $X(t)=(-1)^{N(t)}$ , then  $\{X(t)\}$  is called a semi random telegraph signal process.

5. Define random process. (N/D 2013) (N/D 2014)

**Solution:**

A random process is a collection of RVs  $\{X(s,t)\}$  that are functions of a real variable, namely time  $t$  where  $s \in S$  (sample space) and  $t \in T$  (parameter set or index set).

6. State the properties of an ergodic process. (M/J 2014)

**Solution:**

$\{X(t)\}$  is ergodic if all its statistics can be determined from a single function  $X(t, \omega)$  of the process.

7. Explain any two application of binomial process. (M/J 2014)

**Solution:**

(2) Binomial process is a Markov process

(3) Since  $S_n$  is a binomial random variable,  $P[S_n = m] = nC_m p^m (1-p)^{n-m}$

(4) Expected value of a binomial process is  $np$  and its variance is  $np(1-p)$

8. Define wide sense stationary process. (M/J2013)(A/M 2010) (M/J 2012)(A/M 2016)(A/M2017)

**Solution:**

A random process  $\{X(t)\}$  with finite first and second order moments is called a weakly stationary process or covariance stationary process or wide-sense stationary process if its mean is a constant and the auto correlation depends only on the time difference. i.e, if  $E[X(t)] = \mu$  and  $E[X(t)X(t-\tau)] = R(\tau)$ .

9. Show that a binomial process is a Markov process. (M/J2013)

**Solution:**

Let  $S_n = X_1 + X_2 + \dots + X_{n-1} + X_n$

$$S_n = S_{n-1} + X_n$$

$$P[S_n = m / S_{n-1} = m] = P[X_n = 0] = 1 - p$$

$$P[S_n = m / S_{n-1} = m - 1] = P[X_n = 1] = p$$

Hence binomial process is a Markov process.

10. Consider the random process  $X(t) = \cos(t + \phi)$ , where  $\phi$  is uniformly distributed in the interval  $-\frac{\pi}{2}$  to  $\frac{\pi}{2}$ . Check whether the process is stationary or not. (N/D 2010)

$$E(X(t)) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t + \phi) d\phi = \frac{1}{\pi} [\sin(t + \phi)]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2 \cos t$$

Hence the process is not stationary.

11. State the postulates of Poisson process. (N/D 2010)(A/M 2011) (A/M 2016)

**Solution:**

The postulates of Poisson process are

- $P[\text{1 occurrence in } (t, t + \Delta t)] = \lambda \Delta t + O(\Delta t)$
- $P[\text{0 occurrence in } (t, t + \Delta t)] = 1 - \lambda \Delta t + O(\Delta t)$
- $P[\text{2 or more occurrences in } (t, t + \Delta t)] = O(\Delta t)$
- $X(t)$  is independent of the number of occurrences of the event in any interval prior to and after the interval  $(0, t)$
- The probability that the event occurs a specified number of times in  $(t_0, t_0 + t)$  depends only on  $t$ , but not on  $t_0$

12. Prove that a first order stationary process has a constant mean. (A/M 2011)

**Solution:**

$f[x(t)] = f[x(t+h)]$  as the process is stationary.

$$E[X(t)] = \int x(t) f[x(t)] dt$$

$$E[X(t+h)] = \int x(t+h) f[x(t+h)] d(t+h)$$

$$t+h = u \Rightarrow d(t+h) = du$$

$$\begin{aligned} \text{Put} \quad &= \int x(u) f[x(u)] du \\ &= E[X(u)] \end{aligned}$$

$$\therefore E[X(t+h)] = E[X(t)]$$

Therefore,  $E[X(t)]$  is independent of  $t$ .

$\therefore E[X(t)]$  is a constant

13. Define a stationary process. (N/D 2004)(M/J 2010)

**Solution:**

If certain probability distribution or averages do not depend on  $t$ , then the random process  $\{X(t)\}$  is called a stationary process.

14. State the four types of stochastic processes. (M/J 2010)(A/M 2010) (A/M 2004)

**Solution:**

The four types of stochastic processes are  
Discrete random sequence

Continuous random sequence  
Discrete random process  
Continuous random process

15. Give an example for a continuous time random process. (N/D 2004)

**Solution:**

If  $X(t)$  represents the maximum temperature at a place in the interval  $(0, t)$ ,  $\{X(t)\}$  is a continuous random process.

16. Define strict sense stationary process. (N/D 2004) (N/D 2012)

**Solution:**

A random process is called a strict sense stationary process or strongly stationary process if all its finite dimensional distributions are invariant under translation of time parameter.

17. Give an example of an ergodic process. (A/M 2004)

**Solution:**

A Markov chain finite state space.

A stochastic process  $X(t)$  is ergodic if its time average tends to the ensemble average as  $T \rightarrow \infty$

18. When is a random process or stochastic process said to be ergodic? (N/D 2005)

**Solution:**

A random process  $\{X(t)\}$  is said to be ergodic, if its ensemble averages are equal to appropriate time averages.

19. State the properties of an ergodic process. (N/D 2004)

**Solution:**

$\{X(t)\}$  is ergodic if all its statistics can be determined from a single function  $X(t, \omega)$  of the process.

20. Prove that the difference of two independent Poisson processes is not a Poisson process. (M/J 2010) (A/M 2010)

**Solution:**

$$\begin{aligned} \text{Let } X(t) &= X_1(t) - X_2(t) \\ E[X(t)] &= E[X_1(t) - X_2(t)] \\ &= E[X_1(t)] - E[X_2(t)] \\ &= \lambda_1 t - \lambda_2 t \\ &= (\lambda_1 - \lambda_2)t \\ E[X^2(t)] &= E[(X_1(t) - X_2(t))^2] \end{aligned}$$

21. Define Markov chain and one – step transition probability. (A/M 2010) (A/M 2016)

**Solution:**

$$\text{If } \forall n, P\left[X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \dots, X_0 = a_0\right] = P\left[X_n = a_n / X_{n-1} = a_{n-1}\right]$$

then the process  $\{X_n\}, n=0,1,2, \dots$  is called a Markov chain.

The conditional probability  $P\left[X_n = a_j / X_{n-1} = a_i\right]$  is called the one step transition probability from state  $a_i$  to state  $a_j$  at the  $n^{\text{th}}$  step.

22. Give an example of a Markov process. (N/D 2003)

**Solution:**

Poisson process is a Markov process. Therefore, number of arrivals in  $(0,t)$  is a Poisson process and hence a Markov process.

23. Define Poisson process. (N/D 2010)

**Solution:**

If  $X(t)$  represents the number of occurrences of a certain event in  $(0,t)$ , then the discrete process  $\{X(t)\}$  is called the Poisson process.

24. If the transition probability matrix of a markov chain is  $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$  find the steady-state

distribution of the chain.

**Solution:**

Let  $\pi = (\pi_1, \pi_2)$  be the limiting form of the state probability distribution on stationary state distribution of the markov chain.

By the property of  $\pi$ ,  $\pi P = \pi$

$$\text{i.e., } (\pi_1, \pi_2) \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\pi_1, \pi_2)$$

$$\frac{1}{2} \pi_2 = \pi_1 \text{----- (1)}$$

$$\pi_1 + \frac{1}{2} \pi_2 = \pi_2 \text{----- (2)}$$

Equation (1) & (2) are one and the same.

Consider (1) or (2) with  $\pi_1 + \pi_2 = 1$ , since  $\pi$  is a probability distribution.

$$\pi_1 + \pi_2 = 1$$

$$\text{Using (1), } \frac{1}{2} \pi_2 + \pi_2 = 1$$

$$\frac{3\pi_2}{2} = 1$$

$$\pi_2 = \frac{2}{3}$$



$$\pi_1 = 1 - \pi_2 = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\pi_2 = 1 - \pi_1 = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\therefore \pi_1 = \frac{1}{3} \text{ \& } \pi_2 = \frac{2}{3}.$$

25. Consider the random process,  $X(t) = \cos(\omega_0 t + \theta)$  where  $\theta$  is uniformly distributed in the interval  $-\pi$  to  $\pi$ . Check whether the process is stationary or not. (A/M 2004)

**Solution:**

$$E[X(t)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\omega_0 t + \theta) d\theta = \frac{1}{2\pi} [\sin(\omega_0 t + \pi) - \sin(\omega_0 t - \pi)] = \frac{1}{2\pi} [-\sin(\omega_0 t) + \sin(\omega_0 t)] = 0$$

$$E[X^2(t)] = \frac{1}{4\pi} \left[ \frac{\theta - 2\sin(\omega_0 t + \theta)}{2} \right]_{-\pi}^{\pi} = \frac{1}{2}$$

**UNIT III**  
**PART B**

1. Examine whether  $X(t) = A \cos \lambda t + B \sin \lambda t$  where A and B are random variables such that  $E(A) = E(B) = 0$ ;  $E(A^2) = E(B^2) = E(AB) = 0$  is wide sense stationary. (A/M 2015) (MA6451 A/M2015)
2. Find the autocorrelation function of the poisson process. (A/M 2015)
3. Suppose X(t) is a normal process with mean  $\mu(t) = 3$ ,  $C_x(t_1, t_2) = 4 e^{-0.2|t_1 - t_2|}$ . Find  $P(X(5) \leq 2)$  and  $P(|X(8) - X(5)| \leq 1)$ . (A/M 2015)
4. Define a random telegraph process. Show that it is a covariance stationary process. (A/M 2015) (A/M 2016)
5. There are 2 white marbles in urn A and 3 red marbles in urn B. At each step of the process, a marble is selected from each urn and the 2 marbles selected are inter changed. Let the state  $a_i$  of the system be the number of red marbles in A after  $i$  changes. What is the probability that there are 2 red marbles in A after 3 steps? In the long run, what is the probability that there are 2 red marbles in urn A? (MA6451 A/M2015)
6. A radioactive source emits particles at a rate of 5 per minute in accordance with poisson process. Each particle emitted has a probability 0.6 of being recorded. Find the probability that 10 particles are recorded in 4 minute period. (MA6451 A/M2015)
7. Check if a random telegraph signal process is WSS. (MA6451 A/M2015) (N/D2013)
8. If  $\{X(t)\}$  is a Gaussian process with  $\mu(t) = 10$  &  $C(t_1, t_2) = 16 e^{-|t_1 - t_2|}$ . Find the probability that (i)  $X(10) \leq 8$  and (ii)  $|X(10) - X(6)| \leq 4$ . (N/D2013) (N/D 2014) (A/M 2011)
9. Prove that sum of two independent Poisson processes is poisson process. (N/D2013) (A/M 2016)
10. The probability distribution of the process  $\{X(t)\}$  is given by

$$P[X(t) = n] = \begin{cases} \frac{(at)^{n-1}}{(1+at)^{n+1}}, & n = 1, 2, 3, \dots \\ \frac{at}{1+at} & n = 0 \end{cases}$$

Show that it is not stationary. (N/D 2014) (M/J 2014) (N/D 2010) (N/D 2012)

11. If the  $2n$  random variables  $A_r$  and  $B_r$  are uncorrelated with zero mean and  $E(A_r^2) = E(B_r^2) = \sigma_r^2$ . Find the mean and autocorrelation of the process  $X(t) = \sum_{r=1}^n A_r \cos \omega_r t + B_r \sin \omega_r t$ . (N/D 2014) (N/D 2013)
12. Define semi-random telegraph signal process and random telegraph signal process and prove also that the former is evolutionary and the latter is WSS. (N/D 2014) (M/J2013)
13. If the WSS process  $\{X(t)\}$  is given by  $X(t) = 10 \cos(100t + \theta)$  where  $\theta$  is uniformly distributed over  $(-\pi, \pi)$  prove that  $\{X(t)\}$  is correlation ergodic. (M/J 2014) (N/D 2010) (N/D 2012)
14. Mention any three properties each of autocorrelation and of cross correlation functions of wide sense stationary process. (M/J2013)
15. A random process  $X(t)$  defined by  $X(t) = A \cos t + B \sin t$ ;  $-\infty < t < \infty$  where  $A$  and  $B$  are independent random variables each of which has a value  $-2$  with probability  $1/3$  and a value  $1$  with probability  $2/3$ . Show that  $X(t)$  is a WSS. (M/J 2013) (A/M 2011)
16. If the process  $\{X(t); t \geq 0\}$  is a Poisson process with parameter  $\lambda$ , obtain  $P[X(t) = n]$ . Is the process first order stationary? (N/D 2010)
17. Prove that a random telegraph signal process  $Y(t) = \alpha X(t)$  is a Wide Sense Stationary Process when  $\alpha$  is a random variable which is independent of  $X(t)$ , assume value  $-1$  and  $+1$  with equal probability and  $R_{XX}(t_1, t_2) = e^{-2\lambda|t_1 - t_2|}$ . (N/D 2010) (N/D 2012)
18. Prove that the random process  $X(t) = A \cos(\omega t + \theta)$  where  $\omega$  are constants,  $A$  is a random variable with zero mean and variance one and  $\theta$  is uniformly distributed on the interval  $(0, 2\pi)$ . Assume that the random variable  $A$  and  $\theta$  are independent. Is  $X(t)$  is mean-ergodic process? (A/M 2011) (A/M 2016)
19. Prove that the interval between two successive occurrences of a Poisson process with parameter  $\lambda$  has an exponential distribution with mean  $\frac{1}{\lambda}$ . (A/M 2011) (A/M 2016)
20. If the process  $\{X(t)\}$  is a Poisson process with parameter  $\lambda$ . Obtain  $P(X(t)=n)$ . Is the process first order stationary? (N/D 2012)
21. Suppose that a customer arrives at a bank according to Poisson process with a mean rate of 3 per minute. Find the probability that during a time interval of 2 min.
- (i) exactly 4 customers arrive and (2) more than 4 customers arrive. (N/D 2013)

**UNIT IV**  
**CORRELATION AND SPECTRAL DENSITIES**

**PART A**

1. Define power spectral density function of stationary random process. (A/M 2015)

**Solution:**

If  $\{X(t)\}$  is a stationary process with autocorrelation function  $R(\tau)$ , then the Fourier transform of  $R(\tau)$  is called the power spectral density function of  $\{X(t)\}$  and denoted as  $S(\omega)$  or  $S_{XX}(\omega)$ .

$$i.e. \quad S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau.$$

2. If  $R_{XX}(\tau) = \frac{25\tau^2 + 36}{6.25\tau^2 + 4}$ . Find the mean and variance of X. (A/M 2015) (N/D 2016)

**Solution:**

$$\begin{aligned} \text{Given } R_{XX}(\tau) &= \frac{25\tau^2 + 36}{6.25\tau^2 + 4} \\ \mu_x^2 &= \lim_{\tau \rightarrow \infty} R(\tau) \\ &= \lim_{\tau \rightarrow \infty} \left( \frac{25\tau^2 + 36}{6.25\tau^2 + 4} \right) \\ &= \lim_{\tau \rightarrow \infty} \left( \frac{\tau^2 \left( 25 + \frac{36}{\tau^2} \right)}{\tau^2 \left( 6.25 + \frac{4}{\tau^2} \right)} \right) \\ &= \frac{25 + 0}{6.25 + 0} = 4 \end{aligned}$$

$$\text{Mean} = \mu_x = E[X(t)] = 2$$

$$\begin{aligned} E[X^2(t)] &= R_{XX}(0) \\ &= \frac{25(0) + 36}{6.25(0) + 4} \\ &= \frac{36}{4} = 9 \end{aligned}$$

$$\begin{aligned} \text{Variance} &= E[X^2(t)] - (E[X(t)])^2 \\ &= 9 - (2)^2 = 9 - 4 = 5. \end{aligned}$$

3. State any two properties of cross correlation function. (MA6451 A/M2015) (N/D 2014) (N/D 2010) (N/D 2016)

**Solution:**

Properties of cross correlation function are:

- i.  $R_{XY}(-\tau) = R_{YX}(\tau)$ .
- ii. If the process  $\{X(t)\}$  and  $\{Y(t)\}$  are orthogonal, then  $R_{XY}(\tau) = 0$ .
- iii. If the process  $\{X(t)\}$  and  $\{Y(t)\}$  are independent, then  $R_{XY}(\tau) = E[X(t)]E[Y(t-\tau)]$ .

4. Define cross correlation function and state any two of its properties. (M/J 2014)

**Solution:**

If the process  $\{X(t)\}$  and  $\{Y(t)\}$  are jointly wide sense stationary, then  $E[X(t)Y(t-\tau)]$  is a function of  $\tau$ , denoted by  $R_{XY}(\tau)$ . This function  $R_{XY}(\tau)$  is called the cross correlation function of the process  $\{X(t)\}$  and  $\{Y(t)\}$ .

Properties of cross correlation function are:

$$R_{XY}(-\tau) = R_{YX}(\tau).$$

If the process  $\{X(t)\}$  and  $\{Y(t)\}$  are orthogonal, then  $R_{XY}(\tau) = 0$ .

If the process  $\{X(t)\}$  and  $\{Y(t)\}$  are independent, then  $R_{XY}(\tau) = E[X(t)]E[Y(t-\tau)]$ .

5. The Power spectral density of a random process  $\{X(t)\}$  is given by

$$S_{XX}(\omega) = \begin{cases} \pi, & |\omega| < 1 \\ 0, & \text{elsewhere} \end{cases} \quad \text{Find its autocorrelation function. (MA6451 A/M2015)}$$

**Solution:**

$$\begin{aligned} R_{XX}(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{i\omega\tau} d\omega \\ &= \frac{1}{2\pi} \int_{-1}^1 \pi e^{i\omega\tau} d\omega \\ &= \frac{1}{2} \left[ \frac{e^{i\omega\tau}}{i\tau} \right]_{-1}^1 \\ &= \frac{1}{2} \left[ \frac{e^{i\tau} - e^{-i\tau}}{i\tau} \right] \\ &= \frac{1}{\tau} \left[ \frac{e^{i\tau} - e^{-i\tau}}{2i} \right] = \frac{\sin\tau}{\tau} \end{aligned}$$

6. Define power spectral density function. (N/D 2013) (N/D 2016)

**Solution:**

If  $\{X(t)\}$  is a stationary process with autocorrelation function  $R(\tau)$ , then the Fourier transform of  $R(\tau)$  is called the power spectral density function of  $\{X(t)\}$  and denoted as  $S(\omega)$  or  $S_{XX}(\omega)$ .

$$i.e. \quad S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau.$$

7. State Wiener Khinchine theorem. (N/D 2013) (N/D 2014) (N/D 2016)

**Solution:**

If  $X_T(\omega)$  is the Fourier transform of the truncated random process defined as  $X_T(\omega) = \begin{cases} X(t) & , \text{for } |t| \leq T \\ 0 & , \text{for } |t| > T \end{cases}$  where  $\{X(t)\}$  is a real WSS process with power spectral density function  $S(\omega)$ , then  $S(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} E[|X_T(\omega)|^2]$ .

8. State any two properties of auto correlation function. (N/D 2014)

**Solution:**

i.  $R(\tau)$  is an even function of  $\tau$ .

ii. If  $R(\tau)$  is the autocorrelation function of a stationary process  $\{X(t)\}$  with no periodic component, then  $\lim_{\tau \rightarrow \infty} R(\tau) = \mu_x^2$ , provided the limit exists.

9. The autocorrelation function of a stationary process is  $R_{XX}(\tau) = 25 + \frac{4}{1 + 6\tau^2}$ . Find the mean and variance of the process. (N/D 2014) (M/J 2014) (N/D 2016)

**Solution:**

$$\begin{aligned} \text{Given } R_{XX}(\tau) &= 25 + \frac{4}{1 + 6\tau^2} \\ \mu_x^2 &= \lim_{\tau \rightarrow \infty} R(\tau) \\ &= \lim_{\tau \rightarrow \infty} \left( 25 + \frac{4}{1 + 6\tau^2} \right) \\ &= 25 + \lim_{\tau \rightarrow \infty} \left( \frac{4}{1 + 6\tau^2} \right) \\ &= 25 + 0 = 25 \end{aligned}$$

$$\text{Mean} = \mu_x = E[X(t)] = 5$$

$$\begin{aligned}
E[X^2(t)] &= R_{XX}(0) \\
&= 25 + \frac{4}{1+6(0)} \\
&= 25 + 4 = 29
\end{aligned}$$

$$\begin{aligned}
\text{Variance} &= E[X^2(t)] - (E[X(t)])^2 \\
&= 29 - (5)^2 = 29 - 25 = 4
\end{aligned}$$

10. Define cross correlation function of  $X(t)$  and  $Y(t)$ . When do you say that they are independent? (M/J2013)(A/M 2016)

**Solution:**

If the process  $\{X(t)\}$  and  $\{Y(t)\}$  are jointly wide sense stationary, then  $E[X(t)Y(t-\tau)]$  is a function of  $\tau$ , denoted by  $R_{XY}(\tau)$ . This function  $R_{XY}(\tau)$  is called the cross correlation function of the process  $\{X(t)\}$  and  $\{Y(t)\}$ .

If the process  $\{X(t)\}$  and  $\{Y(t)\}$  are independent, then  $R_{XY}(\tau) = E[X(t)]E[Y(t-\tau)]$ .

11. Find the variance of the stationary process  $X(t)$  whose autocorrelation function is given by  $R_{XX}(\tau) = 2 + 4e^{-2|\tau|}$ . (N/D 2010) (N/D 2012) (A/M 2016)

**Solution:**

$$\begin{aligned}
\text{Given } R_{XX}(\tau) &= 2 + 4e^{-2|\tau|} \\
\mu_x^2 &= \lim_{\tau \rightarrow \infty} R(\tau) \\
&= \lim_{\tau \rightarrow \infty} (2 + 4e^{-2|\tau|}) \\
&= 2 + \lim_{\tau \rightarrow \infty} (4e^{-2|\tau|}) \\
&= 2 + 0 = 2
\end{aligned}$$

$$\text{Mean} = \mu_x = E[X(t)] = \sqrt{2}$$

$$\begin{aligned}
E[X^2(t)] &= R_{XX}(0) \\
&= 2 + 4e^{-2(0)} \\
&= 2 + 4 = 6
\end{aligned}$$

$$\begin{aligned}
\text{Variance} &= E[X^2(t)] - (E[X(t)])^2 \\
&= 6 - (\sqrt{2})^2 = 6 - 2 = 4
\end{aligned}$$

12. Prove that for a WSS process  $\{X(t)\}$ ,  $R_{XX}(\tau) = R_{XX}(-\tau)$ . (A/M 2011)

**Solution:**

$$R_{XX}(\tau) = E[X(t)X(t-\tau)]$$

$$\begin{aligned}
 R_{XX}(-\tau) &= E[X(t)X(t+\tau)] \\
 &= E[X(t+\tau)X(t)] = R_{XX}(\tau)
 \end{aligned}$$

Therefore  $R(\tau)$  is an even function of  $\tau$ .

13. The autocorrelation function of a stationary process is  $R_{XX}(\tau) = 16 + \frac{9}{1+6\tau^2}$ . Find the mean and variance of the process. (A/M 2011) (M/J 2012)

**Solution:**

$$\begin{aligned}
 \text{Given } R_{XX}(\tau) &= 16 + \frac{9}{1+6\tau^2} \\
 \mu_x^2 &= \lim_{\tau \rightarrow \infty} R(\tau) \\
 &= \lim_{\tau \rightarrow \infty} \left( 16 + \frac{9}{1+6\tau^2} \right) \\
 &= 16 + \lim_{\tau \rightarrow \infty} \left( \frac{9}{1+6\tau^2} \right) \\
 &= 16 + 0 = 16
 \end{aligned}$$

$$\begin{aligned}
 \text{Mean } = \mu_x &= E[X(t)] = 4 \\
 E[X^2(t)] &= R_{XX}(0) \\
 &= 16 + \frac{9}{1+6(0)} \\
 &= 16 + 9 = 25 \\
 \text{Variance} &= E[X^2(t)] - (E[X(t)])^2 \\
 &= 25 - (4)^2 = 25 - 16 = 9
 \end{aligned}$$

14. The autocorrelation function of a stationary process is  $R_{XX}(\tau) = 18 + \frac{2}{6+\tau^2}$ . Find the mean and variance of the process. (A/M 2006)

**Solution:**

$$\begin{aligned}
 \text{Given } R_{XX}(\tau) &= 18 + \frac{2}{6+\tau^2} \\
 \mu_x^2 &= \lim_{\tau \rightarrow \infty} R(\tau) \\
 &= \lim_{\tau \rightarrow \infty} \left( 18 + \frac{2}{6+\tau^2} \right) \\
 &= 18 + \lim_{\tau \rightarrow \infty} \left( \frac{2}{6+\tau^2} \right) \\
 &= 18 + 0 = 18
 \end{aligned}$$

$$\text{Mean } = \mu_x = E[X(t)] = \sqrt{18}$$



$$\begin{aligned}
E[X^2(t)] &= R_{XX}(0) \\
&= 18 + \frac{2}{6+0} \\
&= \frac{55}{3} \\
\text{Variance} &= E[X^2(t)] - (E[X(t)])^2 \\
&= \frac{55}{3} - (\sqrt{18})^2 = \frac{1}{3}
\end{aligned}$$

15. Find the variance of the stationary process  $X(t)$  whose autocorrelation function is given by  $R_{XX}(\tau) = 9 + 2e^{-2|\tau|}$ . (A/M 2003) (NOV/D 2016)

**Solution:**

$$\begin{aligned}
\text{Given } R_{XX}(\tau) &= 9 + 2e^{-2|\tau|} \\
\mu_x^2 &= \lim_{\tau \rightarrow \infty} R(\tau) \\
&= \lim_{\tau \rightarrow \infty} 9 + 2e^{-2|\tau|} \\
&= 9 + \lim_{\tau \rightarrow \infty} (2e^{-2|\tau|}) \\
&= 9 + 0 = 9 \\
\text{Mean } &= \mu_x = E[X(t)] = 3 \\
E[X^2(t)] &= R_{XX}(0) \\
&= 9 + 2e^{-2(0)} \\
&= 9 + 2 = 11 \\
\text{Variance} &= E[X^2(t)] - (E[X(t)])^2 \\
&= 11 - (3)^2 = 2
\end{aligned}$$

16. Statistically independent zero mean random processes  $X(t)$  and  $Y(t)$  have autocorrelation functions  $R_{XX}(\tau) = e^{-|\tau|}$  and  $R_{YY}(\tau) = \cos 2\pi\tau$  respectively. Find the autocorrelation function of the sum  $Z(t) = X(t) + Y(t)$ . (A/M 2010)

**Solution:**

$$\begin{aligned}
\text{Given } R_{XX}(\tau) &= e^{-|\tau|}, R_{YY}(\tau) = \cos 2\pi\tau \\
E[X(t)] &= E[Y(t)] = 0 \\
\text{If } Z(t) &= X(t) + Y(t), \text{ then } R_{ZZ}(\tau) = R_{XX}(\tau) + R_{YY}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) \\
R_{XY}(\tau) &= E[X(t)Y(t-\tau)] \text{ [Since the processes are independent]} \\
R_{XY}(\tau) &= (0)(0) \\
&= 0 \\
\text{Similarly } R_{YX}(\tau) &= 0
\end{aligned}$$

$$\begin{aligned}\therefore R_{ZZ}(\tau) &= e^{-|\tau|} + \cos 2\pi\tau + 0 + 0 \\ &= e^{-|\tau|} + \cos 2\pi\tau\end{aligned}$$

17. If  $R(\tau) = e^{-2\lambda|\tau|}$  is the auto correlation function of a random process  $X(t)$ , obtain the spectral density of  $X(t)$ . (N/D 2004)

**Solution:**

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau = \int_{-\infty}^{\infty} e^{-2\lambda|\tau|} e^{-i\omega\tau} d\tau = 2 \int_0^{\infty} e^{-2\lambda\tau} \cos \omega\tau d\tau = \frac{4\lambda}{4\lambda^2 + \omega^2}.$$

18. If the power spectral density of a WSS process is give by  $S(\omega) = \begin{cases} \frac{b}{a}(a - |\omega|) & , |\omega| < a \\ 0 & , |\omega| > a \end{cases}$ .

Find the auto correlation. (A/M 2010)(N/D 2007)( NOV/D 2016)

**Solution:**

$$\begin{aligned}R(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega \\ &= \frac{1}{2\pi} \left[ \int_{-\infty}^{-a} S(\omega) e^{i\omega\tau} d\omega + \int_{-a}^a S(\omega) e^{i\omega\tau} d\omega + \int_a^{\infty} S(\omega) e^{i\omega\tau} d\omega \right] \\ &= \frac{1}{2\pi} \left[ \int_{-a}^a \frac{b}{a}(a - |\omega|) e^{i\tau\omega} d\omega \right] \\ &= \frac{b}{2\pi a} \int_{-a}^a (a - |\omega|) (\cos \tau\omega + i \sin \tau\omega) d\omega \\ &= \frac{b}{2\pi a} \int_{-a}^a (a - |\omega|) \cos \tau\omega d\omega + i \frac{b}{2\pi a} \int_{-a}^a (a - |\omega|) \sin \tau\omega d\omega \\ &= \frac{b}{2\pi a} 2 \int_0^a (a - \omega) \cos \tau\omega d\omega + i \frac{b}{2\pi a} (0) \\ &= \frac{b}{\pi a} \int_0^a (a - \omega) \cos \tau\omega d\omega \\ &= \frac{b}{\pi a} \left[ (a - \omega) \frac{\sin \tau\omega}{\tau} - \frac{\cos \tau\omega}{\tau^2} \right]_0^a \\ &= \frac{b}{\pi a} \left[ \left( 0 - \frac{\cos a\tau}{\tau^2} \right) - \left( 0 - \frac{1}{\tau^2} \right) \right] \\ &= \frac{b}{\pi a} \left( \frac{1}{\tau^2} - \frac{\cos a\tau}{\tau^2} \right) \\ R(\tau) &= \frac{b}{\pi a \tau^2} (1 - \cos a\tau).\end{aligned}$$

19. The power spectral density function of a zero mean wide sense stationary process  $\{X(t)\}$  is

$$\text{given by } S(\omega) = \begin{cases} 1 & ; |\omega| < \omega_0 \\ 0 & ; \text{Elsewhere} \end{cases} . \text{ Find } R(\tau) . (\text{M/J 2007})$$

**Solution:**

$$\begin{aligned} R(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega \\ &= \frac{1}{2\pi} \left[ \int_{-\infty}^{-\omega_0} S(\omega) e^{i\omega\tau} d\omega + \int_{-\omega_0}^{\omega_0} S(\omega) e^{i\omega\tau} d\omega + \int_{\omega_0}^{\infty} S(\omega) e^{i\omega\tau} d\omega \right] \\ &= \frac{1}{2\pi} \left[ \int_{-\omega_0}^{\omega_0} 1 \cdot e^{i\omega\tau} d\omega \right] \\ &= \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} (\cos\tau\omega + i \sin \tau\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} (\cos\tau\omega) d\omega + i \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} (\sin \tau\omega) d\omega \\ &= \frac{1}{2\pi} 2 \int_0^{\omega_0} (\cos\tau\omega) d\omega + i \frac{1}{2\pi} (0) [\because \text{The 1}^{\text{st}} \text{ integrand is even and the 2}^{\text{nd}} \text{ is odd}] \\ &= \frac{1}{\pi} \left( \frac{\sin \tau\omega}{\tau} \right)_0^{\omega_0} \\ &= \frac{1}{\pi\tau} (\sin \tau\omega_0 - 0) \\ R(\tau) &= \frac{\sin \omega_0\tau}{\pi\tau} . \end{aligned}$$

20. The autocorrelation function of the random telegraph signal process is given by  $R(\tau) = a^2 e^{-2\lambda|\tau|}$ . Determine the power density spectrum of the random telegraph signal. (A/M 2010)

**Solution:**

$$\text{Given } R(\tau) = a^2 e^{-2\lambda|\tau|}$$

$$\begin{aligned}
S(\omega) &= \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau \\
&= \int_{-\infty}^{\infty} a^2 e^{-2\lambda|\tau|} e^{-i\omega\tau} d\tau \\
&= a^2 \int_{-\infty}^{\infty} e^{-2\lambda|\tau|} (\cos\omega\tau - i \sin\omega\tau) d\tau \\
&= a^2 \int_{-\infty}^{\infty} e^{-2\lambda|\tau|} (\cos\omega\tau) d\tau - ia^2 \int_{-\infty}^{\infty} e^{-2\lambda|\tau|} (\sin\omega\tau) d\tau \\
&= 2a^2 \int_0^{\infty} e^{-2\lambda|\tau|} \cos\omega\tau d\tau \\
&= 2a^2 \left[ \frac{e^{-2\lambda\tau}}{(-2\lambda)^2 + \omega^2} (-2\lambda \cos\omega\tau + \omega \sin\omega\tau) \right]_0^{\infty} \\
&= 2a^2 \left[ 0 - \frac{1}{4\lambda^2 + \omega^2} (-2\lambda + 0) \right] \\
S(\omega) &= \frac{4\lambda a^2}{4\lambda^2 + \omega^2}
\end{aligned}$$

21. The power spectral density of the random process  $\{X(t)\}$  is given by  $S(\omega) = \begin{cases} \pi & ; |\omega| < 1 \\ 0 & ; \text{Elsewhere} \end{cases}$ . Find its autocorrelation function. (A/M 2003) (NOV/D 2016)

**Solution:**

$$\begin{aligned}
R(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega \\
&= \frac{1}{2\pi} \left[ \int_{-\infty}^{-1} S(\omega) e^{i\omega\tau} d\omega + \int_{-1}^1 S(\omega) e^{i\omega\tau} d\omega + \int_1^{\infty} S(\omega) e^{i\omega\tau} d\omega \right] \\
&= \frac{1}{2\pi} \left[ \int_{-1}^1 \pi e^{i\omega\tau} d\omega \right] \\
&= \frac{1}{2\pi} \int_{-1}^1 \pi (\cos\tau\omega + i \sin\tau\omega) d\omega \\
&= \frac{1}{2} \int_{-1}^1 (\cos\tau\omega) d\omega + i \frac{1}{2} \int_{-1}^1 (\sin\tau\omega) d\omega \\
&= \frac{1}{2} \int_0^1 (\cos\tau\omega) d\omega + i \frac{1}{2\pi} (0)
\end{aligned}$$

$$\begin{aligned}
&= \left( \frac{\sin \tau \omega}{\tau} \right)_0^1 \\
&= \frac{1}{\tau} (\sin \tau - 0) \\
R(\tau) &= \frac{\sin \tau}{\tau}.
\end{aligned}$$

22. Given the power spectral density  $S_{xx}(\omega) = \frac{1}{4 + \omega^2}$ , find the average power of the process.

(M/J 2006) (M/J 2010)

**Solution:**

$$\begin{aligned}
R(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{4 + \omega^2} e^{i\tau\omega} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{4 + z^2} e^{i\tau z} dz \\
&= \frac{1}{2\pi} 2\pi i \left[ \lim_{z \rightarrow 2i} (z - 2i) \frac{e^{i\tau z}}{(z + 2i)(z - 2i)} \right] \\
&= i \left[ \frac{e^{i\tau \cdot 2i}}{(2i + 2i)} \right] \\
&= i \left[ \frac{e^{-2\tau}}{4i} \right] \\
R(\tau) &= \frac{e^{-2\tau}}{4}
\end{aligned}$$

$$\text{Average power} = R(0) = \frac{e^{(0)}}{4} = \frac{1}{4}$$

23. Find the power spectral density of a WSS process with autocorrelation function  $R(\tau) = e^{-\alpha\tau^2}$ . (N/D 2008)(M/J 2009)(M/J 2010) (A/M 2016)

**Solution:**

Given  $R(\tau) = e^{-\alpha\tau^2}$

$$\begin{aligned}
S(\omega) &= \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau \\
&= \int_{-\infty}^{\infty} e^{-\alpha\tau^2} e^{-i\omega\tau} d\tau
\end{aligned}$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} e^{-\alpha\left(\tau^2 + \frac{i\omega\tau}{\alpha}\right)} d\tau \\
&= \int_{-\infty}^{\infty} e^{-\alpha\left(\tau^2 + \frac{i\omega\tau}{\alpha} + \left(\frac{i\omega}{2\alpha}\right)^2 - \left(\frac{i\omega}{2\alpha}\right)^2\right)} d\tau \\
&= \int_{-\infty}^{\infty} e^{-\alpha\left(\tau + \frac{i\omega}{2\alpha}\right)^2} e^{-\alpha\left(\frac{\omega^2}{4\alpha^2}\right)} d\tau \\
&= e^{-\frac{\omega^2}{4\alpha}} \int_{-\infty}^{\infty} e^{-\alpha\left(\tau + \frac{i\omega}{2\alpha}\right)^2} d\tau
\end{aligned}$$

$$\text{Put } \sqrt{\alpha}\left(\tau + \frac{i\omega}{2\alpha}\right) = x \Rightarrow \sqrt{\alpha}d\tau = dx \Rightarrow d\tau = \frac{dx}{\sqrt{\alpha}}$$

When  $\tau = -\infty, x = -\infty$  When  $\tau = \infty, x = \infty$

$$\begin{aligned}
S(\omega) &= e^{-\frac{\omega^2}{4\alpha}} \int_{-\infty}^{\infty} e^{-x^2} \frac{dx}{\sqrt{\alpha}} \\
&= \frac{e^{-\frac{\omega^2}{4\alpha}}}{\sqrt{\alpha}} \int_{-\infty}^{\infty} e^{-x^2} dx \quad \text{i.e.} \quad S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-i\omega\tau} d\tau \\
&= \frac{e^{-\frac{\omega^2}{4\alpha}}}{\sqrt{\alpha}} \sqrt{\pi} \\
&= \sqrt{\frac{\pi}{\alpha}} e^{-\frac{\omega^2}{4\alpha}}
\end{aligned}$$

24. Find the power spectral density of a random signal with autocorrelation function  $e^{-\lambda|\tau|}$ .  
(N/D 2005)

**Solution:**

$$\text{Given } R_{XX}(\tau) = e^{-\lambda|\tau|}$$

$$\begin{aligned}
S(\omega) &= \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau \\
&= \int_{-\infty}^{\infty} e^{-\lambda|\tau|} e^{-i\omega\tau} d\tau \\
&= \int_{-\infty}^{\infty} e^{-\lambda|\tau|} (\cos\omega\tau - i\sin\omega\tau) d\tau \\
&= \int_{-\infty}^{\infty} e^{-\lambda|\tau|} (\cos\omega\tau) d\tau - i \int_{-\infty}^{\infty} e^{-\lambda|\tau|} (\sin\omega\tau) d\tau
\end{aligned}$$

$$\begin{aligned}
&= 2 \int_0^{\infty} e^{-\lambda|\tau|} \cos \omega \tau d\tau \\
&= 2 \left[ \frac{e^{-\lambda\tau}}{(-\lambda)^2 + \omega^2} (-\lambda \cos \omega \tau + \omega \sin \omega \tau) \right]_0^{\infty} \\
&= 2 \left[ 0 - \frac{1}{\lambda^2 + \omega^2} (-\lambda + 0) \right] \\
S(\omega) &= \frac{2\lambda}{\lambda^2 + \omega^2}
\end{aligned}$$

25. State the properties of the cross spectral density function. (M/J 2012)(A/M2017)

**Solution:**

- i)  $S_{XY}(\omega) = S_{YX}(-\omega) = S_{YX}^*(\omega)$
- ii)  $\text{Re}[S_{XY}(\omega)]$  and  $\text{Re}[S_{YX}(\omega)]$  are even function of  $\omega$
- iii)  $\text{Im}[S_{XY}(\omega)]$  and  $\text{Im}[S_{YX}(\omega)]$  are odd function of  $\omega$
- iv)  $S_{XY}(\omega) = 0$  and  $S_{YX}(\omega) = 0$  if  $X(t)$  and  $Y(t)$  are orthogonal.

26. Prove that for a WSS process  $\{X(t)\}$ ,  $R_{XX}(\tau) = R_{XX}(-\tau)$ . (N/D 2012)

**Solution:**

$$\begin{aligned}
R_{XX}(\tau) &= E[X(t)X(t-\tau)] \\
R_{XX}(-\tau) &= E[X(t)X(t+\tau)] \\
&= E[X(t+\tau)X(t)] = R_{XX}(\tau)
\end{aligned}$$

Therefore  $R(\tau)$  is an even function of  $\tau$ .

**UNIT IV**  
**PART B**

1. Consider two random processes  $X(t) = 3\cos(\omega t + \theta)$  and  $Y(t) = 2\cos\left(\omega t + \theta - \frac{\pi}{2}\right)$  where  $\theta$  is a random variable uniformly distributed in  $(\theta, 2\pi)$ . Prove that  $\sqrt{R_{XX}(0)R_{YY}(0)} \geq |R_{XY}(\tau)|$ . (A/M 2015) (MA6451 A/M2015) (N/D 2012)
2. Find the power spectral density of random signal with auto correlation function  $e^{-\lambda|\tau|}$ . (A/M 2015)
3. A random process defined as  $\{X(t)\} = A\cos \omega t + B\sin \omega t$ , where A & B are the random variables with  $E(A) = E(B) = 0$  and  $E(A^2) = E(B^2)$  &  $E(AB) = 0$ . Prove that X(t) is a strict sense stationary process of order 2. (A/M 2015)
4. The power spectrum of a WSS process  $\{X(t)\}$  is given by  $S(\omega) = \frac{1}{(1+\omega^2)^2}$ . Find the autocorrelation function and average power of the process. (A/M 2015)
5. Find the power spectral density function whose autocorrelation function is given by 
$$R(\tau) = \begin{cases} 1 - \frac{|\tau|}{T} & ; |\tau| \leq T \\ 0 & ; \text{elsewhere} \end{cases} \quad (\text{MA6451 A/M2015})$$
6. Given the power spectral density of a continuous process as  $S_{XX}(\omega) = \frac{\omega^2 + 9}{\omega^4 + 5\omega^2 + 4}$  find the mean square value of the process. (MA6451 A/M2015) (A/M 2016)
7. A stationary process has an autocorrelation function given by  $R_{XX}(\tau) = \frac{25\tau^2 + 36}{6.25\tau^2 + 4}$ . Find the mean value, mean square value and variance of the process. (MA6451 A/M2015)
8. The autocorrelation function of the random telegraph signal process is given by  $R(\tau) = a^2 e^{-2\lambda|\tau|}$ . Determine the power density spectrum of the random telegraph signal. (N/D 2013)



9. If the power spectral density of a WSS process is given by  $S(\omega) = \begin{cases} \frac{b}{a}(a-|\omega|) & , |\omega| < a \\ 0 & , |\omega| > a \end{cases}$ .

Find the autocorrelation function of the process. (N/D 2013) (N/D 2014)

10. The autocorrelation function of the Poisson increment process is given

$$\text{by } R(\tau) = \begin{cases} \lambda^2 & ; |\tau| > t \\ \lambda^2 + \frac{\lambda}{t} \left(1 - \frac{|\tau|}{t}\right) & ; |\tau| \leq t \end{cases} . \text{ Prove the spectral density is given}$$

$$\text{by } S(\omega) = 2\pi\lambda^2\delta(\omega) + \frac{4\lambda \sin^2\left(\frac{\omega t}{2}\right)}{t^2\omega^2} . \text{ (N/D 2013)}$$

11. If the process  $\{X(t)\}$  is defined as  $X(t) = Y(t)Z(t)$  where  $\{Y(t)\}$  and  $\{Z(t)\}$  are independent WSS processes, prove that (i)  $R_{XX}(\tau) = R_{YY}(\tau)R_{ZZ}(\tau)$  (ii)

$$S_{XX}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YY}(\omega) S_{ZZ}(\omega) . \text{ (N/D 2013)}$$

12. A random process defined as  $\{X(t)\} = A\cos \omega t + B\sin \omega t$ , where A & B are the random variables with  $E(A) = E(B) = 0$  and  $E(A^2) = E(B^2)$  &  $E(AB) = 0$ . Find the power spectral density of the process. (N/D 2014)

13. Find the power spectral density of WSS with auto correlation function  $R(\tau) = e^{-\alpha\tau^2}$ . (N/D 2014) A/M 2016)

14. The random binary transmission process  $\{X(t)\}$  is a WSS with zero mean and

$$\text{autocorrelation function is given by } R(\tau) = \begin{cases} 1 - \frac{|\tau|}{T} & ; |\tau| \leq T \\ 0 & ; \text{elsewhere} \end{cases}$$

where T is a constant. Find the mean and variance of the time averages of  $\{X(t)\}$  over (0, T). Is  $\{X(t)\}$  mean ergodic? (N/D 2014)

15. Find the power spectral density of the random binary transmission process whose

$$\text{autocorrelation function is } R(\tau) = \begin{cases} 1 - |\tau| & \text{for } |\tau| \leq 1 \\ 0 & \text{elsewhere} \end{cases} . \text{ (N/D 2010) (N/D 2012)}$$

16. The cross-power spectrum of real Random process  $\{X(t)\}$  and  $\{Y(t)\}$  is given by

$$S_{XY}(\omega) = \begin{cases} a + jb\omega; & |\omega| < 1 \\ 0 & ; \text{elsewhere} \end{cases}. \text{ Find the cross correlation function. (N/D 2010)}$$

17. State and prove Wiener – Khinchine Theorem. (N/D 2010)

18. If  $\{X(t)\}$  and  $\{Y(t)\}$  are two random processes with auto correlation function  $R_{XX}(\tau)$  and  $R_{YY}(\tau)$  respectively then prove that  $|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0)R_{YY}(0)}$ . Establish any two properties of auto correlation function  $R_{XX}(\tau)$ . (N/D 2010) (A/M 2016)

19. State and prove Wiener – Khinchine Theorem and hence find the power spectral density function whose autocorrelation function is given by  $R(\tau) = A_0 \left(1 - \frac{|\tau|}{T}\right)$ ,  $-T \leq \tau \leq T$ . (N/D 2012)

**UNIT V**  
**LINEAR SYSTEMS WITH RANDOM INPUTS**

**PART A**

1. Define linear system with random output. (A/M 2015)(A/M2017)

**Solution:**

A system is a functional relationship between the input  $x(t)$  and the output  $y(t)$ . The functional relationship is written as  $y(t) = f[x(t)]$ .

If  $f[a_1 X_1(t) \pm a_2 X_2(t)] = a_1 f[X_1(t)] \pm a_2 f[X_2(t)]$ , then  $f$  is called a linear system.

2. State any two properties of cross power density spectrum. (A/M 2015)

**Solution:**

i)  $S_{XY}(\omega) = S_{YX}(-\omega) = S_{XY}^*(\omega)$

ii)  $\text{Re}[S_{XY}(\omega)]$  and  $\text{Re}[S_{YX}(\omega)]$  are even function of  $\omega$

iii)  $\text{Im}[S_{XY}(\omega)]$  and  $\text{Im}[S_{YX}(\omega)]$  are odd function of  $\omega$

iv)  $S_{XY}(\omega) = 0$  and  $S_{YX}(\omega) = 0$  if  $X(t)$  and  $Y(t)$  are orthogonal.

3. Prove that  $Y(t) = 2X(t)$  is linear. (MA6451 A/M2015)

**Solution:**

$$[a_1 Y_1(t) \pm a_2 Y_2(t)] = a_1 [2X_1(t)] \pm a_2 [2X_2(t)] = 2a_1 [X_1(t)] \pm 2a_2 [X_2(t)]$$

4. State the relation between input and output of a linear time invariant system. (MA6451 A/M2015)

**Solution:**

If the output  $Y(t)$  of a system is expressed as the convolution of the input  $X(t)$  and a function  $h(t)$  (ie)  $Y(t) = \int_{-\infty}^{\infty} h(u)X(t-u)du$ , then  $h(t)$  is called the system weighting function.

5. Define white noise process. (N/D 2013) (N/D 2014)(N/D2017)

**Solution:**

Let  $X(t)$  be a sample function of a WSS noise process, then  $\{X(t), t \in T\}$  is called the white noise if the power density spectrum of  $\{X(t), t \in T\}$  is constant at all frequencies. (ie)  $S_{NV}(\omega) = \frac{N_0}{2}$  where  $N_0$  is a real positive constant.

6. Define linear time invariant system. (N/D 2013) (M/J2013)

**Solution:**

If  $f[a_1 X_1(t) \pm a_2 X_2(t)] = a_1 f[X_1(t)] \pm a_2 f[X_2(t)]$ , then  $f$  is called a linear system. If  $Y(t+h) = f[X(t+h)]$  where  $Y(t) = f[X(t)]$ , then  $f$  is called a time – invariant system or  $X(t)$  and  $Y(t)$  are said to form a time invariant system

7. The autocorrelation function of a stationary process is  $R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2}$ . Find the mean and variance of the process. (N/D 2014) (M/J 2014)

**Solution:**

$$\begin{aligned} \text{Given } R_{XX}(\tau) &= 25 + \frac{4}{1+6\tau^2} \\ \mu_x^2 &= \lim_{\tau \rightarrow \infty} R(\tau) \\ &= \lim_{\tau \rightarrow \infty} \left( 25 + \frac{4}{1+6\tau^2} \right) \\ &= 25 + \lim_{\tau \rightarrow \infty} \left( \frac{4}{1+6\tau^2} \right) \\ &= 25 + 0 = 25 \end{aligned}$$

$$\text{Mean} = \mu_x = E[X(t)] = 5$$

$$\begin{aligned} E[X^2(t)] &= R_{XX}(0) \\ &= 25 + \frac{4}{1+6(0)} \\ &= 25 + 4 = 29 \end{aligned}$$

$$\begin{aligned} \text{Variance} &= E[X^2(t)] - (E[X(t)])^2 \\ &= 29 - (5)^2 = 29 - 25 = 4 \end{aligned}$$

8. Define a system. When is it called a linear system? (M/J 2014) (A/M 2016)

**Solution:**

A system is a functional relationship between the input  $x(t)$  and the output  $y(t)$ . The functional relationship is written as  $y(t) = f[x(t)]$ .

If  $f[a_1 X_1(t) \pm a_2 X_2(t)] = a_1 f[X_1(t)] \pm a_2 f[X_2(t)]$ , then  $f$  is called a linear system.

8. Define band limited white noise. (M/J 2014) (N/D 2010) (N/D 2010) (N/D 2012)

**Solution:**

Noise having a non-zero and constant spectral density over a finite frequency band and zero elsewhere is called band limited white noise.

$$\text{If } \{N(t)\} \text{ is a band limited white noise, then } S_{NN}(\omega) = \begin{cases} \frac{N_0}{2}, & |\omega| \leq \omega_B \\ 0, & \text{elsewhere} \end{cases}.$$

9. State the convolution form of the output of a linear time invariant system. (M/J2013)

**Solution:**

If  $X(t)$  is the input and  $h(t)$  be the system weighting function and  $Y(t)$  is the output, then  $Y(t) = h(t) * X(t) = \int_{-\infty}^{\infty} h(u) X(t-u) du$

10. If  $Y(t)$  is the output of a linear time invariant system with impulse response  $h(t)$ , then find the cross correlation of the input function  $X(t)$  and output function  $Y(t)$ . (N/D 2010)

**Solution:**

$$\text{If } Y(t) = \int_{-\infty}^{\infty} h(u) X(t-u) du, \text{ then } R_{XY}(\tau) = R_{XX}(\tau) * h(\tau)$$

11. Find the system function, if a linear time invariant system has an impulse function

$$h(t) = \begin{cases} \frac{1}{2c} & \text{for } |t| \leq c \\ 0 & \text{for } |t| > c \end{cases}. \text{ (N/D 2012) (N/D 2010)}$$

**Solution :**

$$\begin{aligned} H(\omega) &= F[h(t)] = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt = \int_{-c}^c \frac{1}{2c} e^{-i\omega t} dt \\ &= \frac{1}{2c} \int_{-c}^c (\cos \omega t - i \sin \omega t) dt = \frac{1}{2c} \int_{-c}^c \cos \omega t dt - i \frac{1}{2c} \int_{-c}^c \sin \omega t dt \\ &= \frac{1}{2c} 2 \int_0^c \cos \omega t dt - i \frac{1}{2c} (0) \text{ [since the first and second integrand are even} \\ &\hspace{15em} \text{and odd functions]} \\ &= \frac{1}{c} \left[ \frac{\sin \omega t}{\omega} \right]_0^c = \frac{1}{\omega c} [\sin c \omega - 0] = \frac{\sin c \omega}{c \omega} \end{aligned}$$

12. State the properties of linear system. (N/D 2003)

**Solution:**

The properties of linear system are

If a system is such that its input  $X(t)$  and its output  $Y(t)$  are related by a convolution integral, then the system is a linear time invariant system.

If the input to a time-invariant, stable linear system is a WSS process, the output will also be a WSS process.

The power spectral densities of the input and output processes in the system are connected by the relation  $S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$ , where  $H(\omega)$  is the Fourier transform of unit impulse response function  $h(t)$ .

13. Describe a linear system with an random input. (A/M 2004)

**Solution:**

We assume that  $X(t)$  represents a sample function of a random process  $\{X(t)\}$ , the system produces an output or response  $Y(t)$  and the ensemble of the output functions forms a random process  $\{Y(t)\}$ . The process  $\{Y(t)\}$  can be considered as the output of the system or transformation  $f$  with  $\{X(t)\}$  as the input the system is completely specified by the operator  $f$ .

26. Given an example of a linear system.(A/M 2005)( A/M 2016)

14.

**Solution:**

Consider the system  $f$  with output  $tx(t)$  for an input signal  $x(t)$ .

i.e .  $y(t) = f[X(t)] = tx(t)$  Then the system is linear.

For any two inputs  $x_1(t), x_2(t)$  the outputs are  $tx_1(t)$  and  $tx_2(t)$  Now

$$f[a_1 x_1(t) + a_2 x_2(t)] = t[a_1 x_1(t) + a_2 x_2(t)] = a_1 tx_1(t) + a_2 tx_2(t)$$

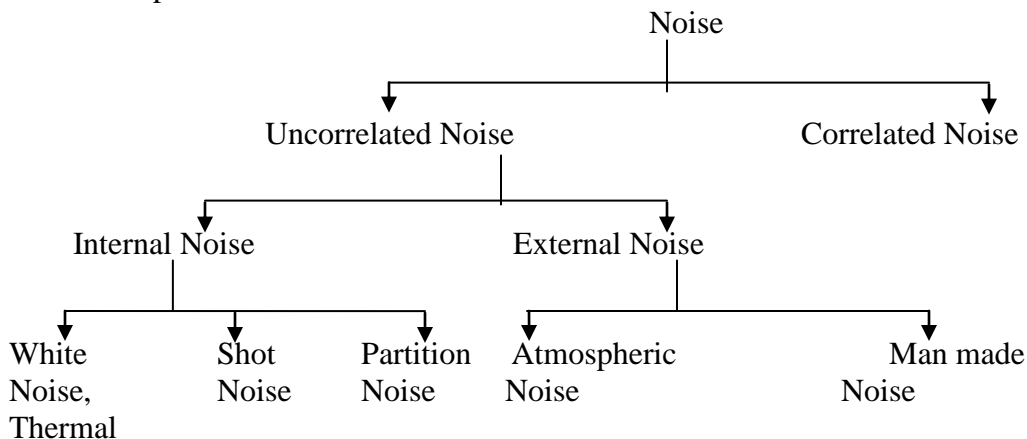
$$= a_1 f(x_1(t)) + a_2 f(x_2(t))$$

$\therefore$  the system is linear.

15. Write a note on noise in communication system. (N/D 2009)

**Solution:**

The term noise is used to designate unwanted signals that tend to disturb the transmission and processing of signal in communication systems and over which we have incomplete control.



## Noise

16. Define band-limited white noise. (N/D 2009)

**Solution:**

Noise with non-zero and constant density over a finite frequency band is called band-limited white noise i.e.,

$$S_{NN}(\omega) = \begin{cases} \frac{N_0}{2}, & |\omega| \leq \omega_B \\ 0, & \text{otherwise} \end{cases}$$

17. Define (a) Thermal Noise (b) White Noise. (N/D 2009)

**Solution:**

(a) Thermal Noise: This noise is due to the random motion of free electrons in a conducting medium such as a resistor.

(or)

Thermal noise is the name given to the electrical noise arising from the random motion of electrons in a conductor.

(b) White Noise(or) Gaussian Noise: The noise analysis of communication systems is based on an idealized form of noise called White Noise.

18. If the input of a linear system is a Gaussian random process, comment about the output random process. (M/J 2012)

**Solution:**

If the input of a linear system is a Gaussian random process, then the output will also be a Gaussian random process.

19. If the power spectral density of white noise is  $\frac{N_0}{2}$ , find its autocorrelation function. (N/D 2004)

**Solution:**

$$\begin{aligned} F\left[\frac{N_0}{2}\delta(\tau)\right] &= \int_{-\infty}^{\infty} \frac{N_0}{2}\delta(\tau)e^{-i\omega\tau}d\tau = \frac{N_0}{2}\int_{-\infty}^{\infty}\delta(\tau)e^{-i\omega\tau}d\tau \\ &= \frac{N_0}{2}e^{(0)} = \frac{N_0}{2} \end{aligned}$$

$$\text{Therefore } R_{NN}(\tau) = F^{-1}\left[\frac{N_0}{2}\right] = \frac{N_0}{2}\delta(\tau).$$

21. If the input to a linear time invariant system is white noise  $\{N(t)\}$ , what is power spectral density function of the output? (M/J 2004) (A/M 2016)

**Solution:**

If the input to a linear time invariant system is white noise  $\{N(t)\}$ , then the

power spectral density of the output  $S_{YY}(\omega)$  is given by  $S_{YY}(\omega) = S_{XX}(\omega)|H(\omega)|^2 = \frac{N_0}{2}|H(\omega)|^2$  where  $\{Y(t)\}$  is the output process and  $H(\omega)$  is the power transfer function.

22. Find the average power or the mean square value of the white noise  $\{N(t)\}$ ? (N/D 2007)

**Solution:**

$$\begin{aligned} \text{Average power} &= E[N^2(t)] = R_{NN}(0) \\ &= \int_{-\infty}^{\infty} S_{NN}(\omega) e^{i0\omega} d\omega = \int_{-\infty}^{\infty} S_{NN}(\omega) d\omega \\ &= \int_{-\infty}^{\infty} \frac{N_0}{2} d\omega \rightarrow \infty \end{aligned}$$

23. State a few properties of band limited white noise. (M/J 2005)

**Solution:**

Properties of band limited white noise are

- (i)  $E[N^2(t)] = \frac{N_0}{2\pi} \omega_B$
- (ii)  $R_{NN}(\tau) = \frac{N_0 \omega_B}{2\pi} \left[ \frac{\sin \omega_B \tau}{\omega_B \tau} \right]$
- (iii)  $N(t)$  and  $N\left(t + \frac{k\pi}{\omega_B}\right)$  are independent, where  $k$  is a non-zero integer.

24. Find the autocorrelation function of the band-limited white noise. (M/J 2006)

**Solution:**

$$\begin{aligned} R_{NN}(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{NN}(\omega) e^{i\tau\omega} d\omega = \frac{1}{2\pi} \int_{-\omega_B}^{\omega_B} \frac{N_0}{2} e^{i\omega\tau} d\omega = \frac{1}{2\pi} \frac{N_0}{2} \int_{-\omega_B}^{\omega_B} (\cos \tau\omega + i \sin \tau\omega) d\omega \\ &= \frac{N_0}{4\pi} \left[ \int_{-\omega_B}^{\omega_B} \cos \tau\omega d\omega + i \int_{-\omega_B}^{\omega_B} \sin \tau\omega d\omega \right] = \frac{N_0}{4\pi} \left[ 2 \int_0^{\omega_B} \cos \tau\omega d\omega + i(0) \right] \\ &\text{[since the first and second integrand are even and odd functions]} \\ &= \frac{N_0}{2\pi} \left[ \frac{\sin \tau\omega}{\tau} \right]_0^{\omega_B} = \frac{N_0}{2\pi\tau} [\sin \tau\omega_B - 0] = \frac{N_0 \omega_B}{2\pi} \frac{\sin \tau\omega_B}{\tau\omega_B} \end{aligned}$$

25. Find the average power of the band – limited white noise. (N/D 2007)

**Solution:**

$$\text{Average power} = E[N^2(t)] = R_{NN}(0)$$



$$\begin{aligned} &= \frac{N_0 \omega_B}{2\pi} \lim_{\tau \rightarrow 0} \left[ \frac{\sin \tau \omega_B}{\tau \omega_B} \right] \\ &= \frac{N_0 \omega_B}{2\pi} (1) \left[ \sin ce \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right] \\ &= \frac{N_0 \omega_B}{2\pi}. \end{aligned}$$

**UNIT V**  
**PART B**

1. A random process  $X(t)$  with  $R_{XX}(\tau) = e^{-2|\tau|}$  is the input to a linear system whose impulse response is  $h(t) = 2e^{-t}, t \geq 0$ . Find cross correlation  $R_{XY}(\tau)$  between the input process  $X(t)$  and the output process  $Y(t)$ . (A/M 2015) (N/D 2012)
  
2. Prove that if the input to a time invariant stable linear system is a wide sense process then the output also is a wide sense stationary process. (A/M 2015) (MA6451 A/M2015) (N/D 2010) (N/D 2012) (A/M 2016)
  
3. Show that the power spectrum  $S_{YY}(\omega)$  of the output of a linear system with system function  $H(\omega)$  is given by  $S_{YY}(\omega) = S_{XX}(\omega) |H(\omega)|^2$  where  $S_{XX}(\omega)$  is the power spectrum of the input. (A/M 2015) (MA6451 A/M2015)
  
4. Let  $Y(t) = X(t) + N(t)$  be a WSS where  $X(t)$  is the actual signal and  $N(t)$  is the zero mean noise process with variance  $\sigma_N^2$ , and independent of  $X(t)$ . Find the power spectral density of  $Y(t)$ . (A/M 2015)
  
5. A circuit has an impulse response given by  $h(t) = \left\{ \frac{1}{T}; 0 \leq t \leq T \right.$ . Express  $S_{XX}(\omega)$  and  $S_{YY}(\omega)$ . (MA6451 A/M2015) (A/M 2016)
  
6. Given  $R_{XX}(\tau) = Ae^{-\alpha|\tau|}$  and  $h(t) = e^{-\beta t}u(t)$  where  $u(t) = \begin{cases} 1, & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$ . Find the spectral density of the output  $Y(t)$ . (MA6451 A/M2015)
  
7. If  $Y(t) = A \cos(\omega_0 t + \theta) + N(t)$ , where  $A$  is a constant,  $\theta$  is a random variable with a uniform distribution in  $(-\pi, \pi)$  and  $\{N(t)\}$  is a band limited Gaussian white noise with a power spectral density  $S_{NN}(\omega) = \begin{cases} \frac{N_0}{2} & , \text{for } |\omega - \omega_0| < \omega_B \\ 0 & , \text{elsewhere} \end{cases}$ . Find the power spectral density of  $\{Y(t)\}$ . Assume that  $N(t)$  and  $\theta$  are independent. (N/D 2013) (A/M 2016)

8.  $X(t)$  is the input voltage to a circuit (system) and  $Y(t)$  is the output voltage.  $\{X(t)\}$  is a stationary random process with  $\mu_x = 0$  and  $R_{XX}(\tau) = e^{-\alpha|\tau|}$ . Find  $\mu_y$ ,  $S_{XX}(\omega)$  and  $R_{YY}(\tau)$ , if the power transfer function is  $H(\omega) = \frac{R}{R+iL\omega}$ ,  $Y(t) = \int_{-\infty}^{\infty} h(\alpha)X(t-\alpha)d\alpha$ . (N/D 2013)

9. Prove that the spectral density of any WSS process is non-negative. (N/D 2013)

10. Check whether the following systems are linear.  $y(t)=tx(t)$  and  $y(t)=x^2(t)$ . (N/D 2014)

The power spectral density of a signal  $x(t)$  is  $S_x(\omega)$  and its power  $P$ . Find the power of the signal  $bx(t)$ . (N/D 2014) (A/M 2011)

11. The linear system is described by the impulse response  $h(t) = \frac{1}{RC}e^{-\frac{t}{RC}}$ . Assume an input signal whose autocorrelation function is  $B\delta(t)$ . Find the autocorrelation mean power of an output. (N/D 2014)

12. If  $Y(t) = A\cos(\omega_0 t + \theta) + N(t)$ , where  $A$  is a constant,  $\theta$  is a random variable with uniform distribution in  $(-\pi, \pi)$  and  $N(t)$  is a band-limited Gaussian white noise with a

power spectral density  $S_{NN}(\omega) = \begin{cases} \frac{N_0}{2}, & \text{for } |\omega - \omega_0| < \omega_B \\ 0, & \text{elsewhere} \end{cases}$ . Find the power spectral density

of  $Y(t)$ . Assume that  $N(t)$  and  $\theta$  are independent. (N/D 2010) (A/M 2011) (N/D 2012)

13. A system has an impulse response function  $h(t) = e^{-\beta t} U(t)$ , find the power spectral density of the output  $Y(t)$  corresponding to the input  $X(t)$ . (N/D 2010) (N/D 2012)

14. If  $X(t)$  is the input voltage to a circuit and  $Y(t)$  is the output voltage.  $\{X(t)\}$  is a stationary random process with  $\mu_x = 0$  and  $R_{XX}(\tau) = e^{-2|\tau|}$ . Find the mean  $\mu_y$  and power spectrum  $S_{YY}(\omega)$  of the output if the system transfer function is given by  $H(\omega) = \frac{1}{\omega + 2i}$ . (N/D 2010)

15. Consider a system with transfer function  $\frac{1}{1+j\omega}$ . An input signal with autocorrelation function  $m\delta(t) + m^2$  is fed as input to the system. Find the mean and mean square value of the output. (A/M 2011)

16. A stationary random process  $X(t)$  having the autocorrelation function  $R_{XX}(\tau) = A\delta(\tau)$  is applied to a linear system at time  $t=0$  where  $f(\tau)$  represent the impulse function. The linear system has the impulse response of  $h(t) = e^{-bt}u(t)$  where  $u(t)$  represent the unit step function. Find  $R_{YY}(\tau)$ . Find the mean and variance of  $Y(t)$ . (A/M 2011)